On density of truth of locally finite logics

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April 21, 2010

$A \subset Form, \quad ||\alpha||$ -length of α

We associate the density $\mu(A)$ with a subset A of formulas as:

$$\mu(A) = \lim_{n\to\infty} \frac{card\,\{\alpha \in A : ||\alpha|| = n\}}{card\,\{\alpha \in Form : ||\alpha|| = n\}}$$

if the appropriate limit exists.

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Densities of some fragments of classical, intuitionistic and modal logics:

- $\mu(Cl^{\rightarrow}_{p,q}) \approx 51.9\%$
- $\mu(Int^{\rightarrow}_{p,q}) \approx 50.43\%$

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- $\mu(Cl^{\rightarrow,\neg}_{p}) \approx 42.3\%$
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Negative examples:

- $\mu(Cl_p^{\leftrightarrow})$,
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- $\mu(Cl_p^{\leftrightarrow,\neg})$,
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Counting formulas - example

Language: p, $\to, \neg$.

Length of formula is defined:

$$\begin{aligned} ||p|| &= 1 \\ ||\phi \to \psi|| &= ||\phi|| + ||\psi|| + 1 \\ ||\neg\phi|| &= ||\phi|| + 1 \end{aligned}$$

F_n - set of formulas of length $n-1$

Number of formulas from F_n is given by the recursion:

$$\begin{aligned} |F_0| &= 0, \;\; |F_1| = 0, \;\; |F_2| = 1 \\ |F_n| &= |F_{n-1}| + \sum_{i=1}^{n-2} |F_i||F_{n-i}|. \end{aligned}$$

Proof: Any formula of the length $n-1$ is either a negation of formula of the length $n-2$ (hence $|F_{n-1}|$) or an implication between some pair of formulas of length $i-1$ and $n-i-1$ (hence $\sum_{i=1}^{n-2} |F_i||F_{n-i}|$).

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Generating functions

Let $(a_n) = (a_0, a_1, a_2, \dots)$ be a sequence of real numbers.

Corresponding formal power series

$$\sum_{n=0}^{\infty} a_n z^n$$

converging uniformly to a function $f_A(z)$ will be called the **generating function**

If the generating function $f_A(z)$ is known we can reconstruct the sequence (a_n) applying the Taylor formula:

$$a_n = \frac{f_A^{(n)}(0)}{n!}.$$

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Example - formulas written with p, $\to, \neg$

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we get that the generating function $f(z) = \sum_{n=0}^{\infty} |F_n| z^n$ fulfils the equation:

$$f(z) = zf(z) + f^2(z) + z^2$$

After solving with boundary condition $f(0) = 0$ we get:

$$f(z) = \frac{1 - z - \sqrt{(z+1)(1-3z)}}{2}$$

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The Drmota-Lalley-Woods theorem

Consider a nonlinear polynomial system, defined by a set of equations $\{y_j = \Phi_j(z, y_1, ..., y_m)\}, \quad 1 \leq j \leq m$ which is a-proper, a-positive, a-irreducible and a-aperiodic. Then

1. All component solutions y_i have the same radius of convergence $\rho < \infty$.
2. There exist functions h_j analytic at the origin such that

$$y_j = h_j(\sqrt{1 - z/\rho}), \quad (z \to \rho^-). \tag{1}$$

3. All y_j have ρ as unique dominant singularity. In that case, the coefficients admit a complete asymptotic expansion of the form:

$$[z^n]y_j(z) \sim \rho^{-n} \left(\sum_{k \geq 1} d_k n^{-1-k/2} \right). \quad (2)$$

[6] Flajolet, P. and Sedgewick, R. *Analitic combinatorics: functional equations, rational and algebraic functions*, INRIA, Number 4103, 2001.

Application of the Drmota-Lalley-Woods theorem

Suppose we have two functions f_T and f_F enumerating the tautologies of some logic and all formulas. Suppose they have the same dominant singularity ρ and there are the suitable constants α_1, α_2, β_1, β_2 such that:

$$f_T(z) = \alpha_1 - \beta_1\sqrt{1 - z/\rho} + O(1 - z/\rho), \quad (3)$$

$$f_F(z) = \alpha_2 - \beta_2\sqrt{1 - z/\rho} + O(1 - z/\rho). \quad (4)$$

Then the *density of truth* is given by:

$$\mu(T) = \lim_{n\to\infty} \frac{[z^n]f_T(z)}{[z^n]f_F(z)} = \frac{\beta_1}{\beta_2}. \quad (5)$$

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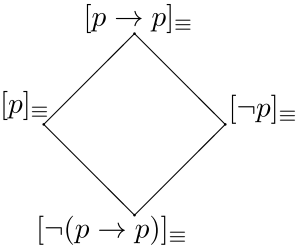
Example: $(Cl_p^{\to,\neg})$

The Lindenbaum algebra of $(Cl_p^{\to,\neg})$ consists of 4 classes:

$$A = [p]_\equiv, \qquad B = [\neg p]_\equiv,$$

$$N = [\neg(p \to p)]_\equiv, \qquad T = [p \to p]_\equiv.$$

Diagram:



From truth-table to system of equations

$\to$	N	A	B	T	$\neg$
N	T	T	T	T	T
A	B	T	B	T	B
B	A	A	T	T	A
T	N	A	B	T	N

$$
\begin{aligned}
f_T(z) &= f_N(z)f(z) + f_A(z)(f_A(z) + f_T(z)) + f_B(z)(f_B(z) + \\
&\quad + f_T(z)) + f_T^2(z) + zf_N(z), \\
f_A(z) &= f_B(z)(f_N(z) + f_A(z)) + f_T(z)f_A(z) + zf_B(z) + z, \\
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Additionally, we know that $f_T + f_A + f_B + f_N = f$

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Additionally, we know that $f_T + f_A + f_B + f_N = f$

Solution

After solving we get f_T, f_A, f_B, f_N. For example:

$$f_T(z) \quad = \quad \frac{\left(24 - \sqrt{2}Z - \sqrt{2}U - 2\sqrt{9 - 90z + 27z^2 + Y + ZU}\right)}{24}$$

where

$$X = \sqrt{(3z+3)(1-3z)},$$
$$Y = \sqrt{3}(3z-3)X,$$
$$Z = \sqrt{9 + 54z - 9z^2 + Y},$$
$$U = \sqrt{9 + 54z + 63z^2 + Y}.$$

All the functions: f_T, f_A, f_B, f_N, f have the same dominant singularity at $z_0 = \frac{1}{3}$.

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Expansions of f and f_T around $z_0 = 1/3$:

$$
\begin{aligned}
f_T(z) &= \alpha + \beta\sqrt{1-3z} + O(1-3z), \\
f(z) &= \frac{2}{3} - \frac{2}{\sqrt{3}}\sqrt{1-3z} + O(1-3z),
\end{aligned}
$$

where

$$\alpha \approx 0.621\ , \qquad \beta \approx -0.489\ .$$

Theorem

$$\mu(Cl_p^{\to,\neg}) \approx \frac{-0.489}{-\frac{2}{\sqrt{3}}} \approx 0.423 .$$

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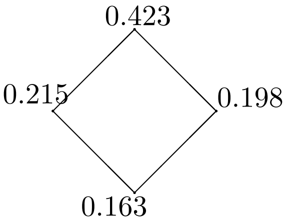

Generalization

Logic L is locally finite (locally tabular) if in a language with a finite number of variables the number of classes of non-equivalent formulas, is also finite.

Let $L_k^{\rightarrow}$ - locally finite logic. We assume that the functor of implication fulfils the following three very general conditions:

(i) $p \rightarrow p \in T_L$,

(ii) for any $\alpha \in Form_k^{\rightarrow}$ it holds: $\alpha \rightarrow (p \rightarrow p) \in T_L$,

(iii) for any $\alpha \in Form_k^{\rightarrow}$ it holds: $(p \rightarrow p) \rightarrow \alpha \in [\alpha]_{\equiv}$.

The conditions hold for the classical and intuitionistic implications as well as for many other implications; e.g. Łukasiewicz's and the strict implication.

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Theorem

Let L be a locally finite purely implicational logic fulfilling the conditions (i)-(iii) in language with k variables. Then the density of truth of L exists.

[7] Z.K., *On the Density of truth of locally finite logics*, JLC, Advanced Access, June 26, 2009.

Proof

- L - locally finite, then Lindenbaum's algebra consists of m equivalence classes $A_1, \ldots A_m$. Let $A_m = T_L$.

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- for each A_i, we may write down a formula describing the way of creating the formulas from the given class. It is the same task as writing the appropriate truth-table.
- After translating each formula into an equation on generating functions, we obtain a system of m equations. By f_i we denote the generating function for the class A_i. Because the conditions (ii) and (iii) hold, the obtained system of equations has to look like:

$$\begin{cases} f_1 = & \ldots + f_m \cdot f_1 + \ldots \\ f_2 = & \ldots + f_m \cdot f_2 + \ldots \\ \ldots = & \ldots\ldots\ldots \\ f_m = & \ldots + (f_1 + f_2 + \ldots + f_m) \cdot f_m + \ldots \end{cases} \tag{6}$$

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- It is easy to prove that the system (6) is a-positive, a-proper, a-irreducible. We should prove that it is a-aperiodic.

  a-aperiodicity: z (not z^2 or z^3...) is the right variable, that means for each f_j there exist three monomials z^a, z^b, and z^c such that $b-a$ and $c-a$ are relatively prime. Then for each generating function $f_j(z)=\sum_{n=0}^{\infty} c_{jn} z^n$ there is some n_0 such that for all $n > n_0$ it holds $c_{jn} \neq 0$.

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The formula $p \to p$ is the shortest tautology (of the length 2). From (ii) we conclude that in the class T_L there are formulas of each length greater than or equal to 2. Then in the expansion $f_m(z) = \sum_{n=2}^{\infty} c_{mn} z^n$ the coefficients $c_{mn} \neq 0$ for $n \geq 2$. Next, from (iii) we conclude that if the shortest formula from A_j has, for instance, the length l, then in the class A_j there are formulas of each length $\geq l+2$. Hence we have $f_j(z) = \sum_{n=l}^{\infty} c_{jn} z^n$, and $c_{jn} \neq 0$ for $n = l$ and $n \geq l+2$. That means that the system of equations (6) is a-aperiodic.

Corollary

Let L be a locally finite logic with implication and other functors as well. Then the density $\mu(L)$ exists.

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Locally finite modal logics with implication

A logic $L \in NEXT(\mathbf{K4})$ is locally finite iff L is of finite depth.

Let us consider the family $\mathbf{K4} \oplus \mathbf{bd_n}$ for each $n \geq 1$, where

$$\mathbf{bd_1} = \Diamond\Box p_1 \to p_1,$$

$$\mathbf{bd_{n+1}} = \Diamond(\Box p_{n+1} \wedge \neg \mathbf{bd_n}) \to p_{n+1}.$$

The logics $\mathbf{K4} \oplus \mathbf{bd_n}$ for each $n \geq 1$ have finite depth.

Theorem

Let $L \in NEXT(\mathbf{K4} \oplus \mathbf{bd_n})$ for any $n \geq 1$. Then its density of truth exists.

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Theorem

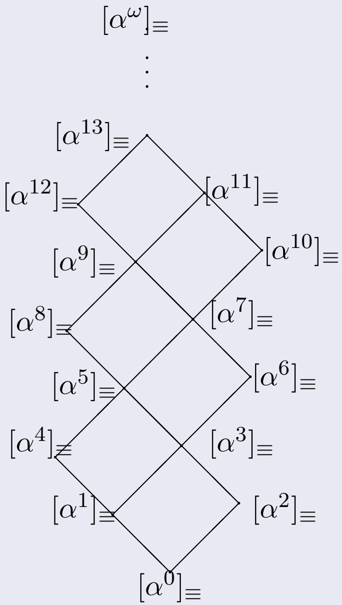
Let $L \in NEXT(\mathbf{K4} \oplus \mathbf{bd_n})$ *for any* $n \geq 1$. *Then its density of truth exists.*

Question

What can we say about logics with implication fulfilling the conditions (i)-(iii) which are not locally finite?

Do they have the density of truth?

Example: $Int_p^{\rightarrow,\neg,\vee}$



Lemma

The density of truth of $Int_p^{\to,\neg,\vee}$ exists and it is estimated as follows:

$$0.7068 \leq \mu(Int_p^{\to,\neg,\vee}) \leq 0.709011$$

Problem

Does $\mu(Int_{p,q}^{\to,\neg,\vee})$ exist?

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