APPLICATIONS OF ALGEBRA
ZAKOPANE 2009

On a finitely axiomatizable Kripke incomplete logic containing KTB

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Brouwerian logic **KTB**

Axioms CL and

$$\begin{aligned} K &:= \Box(p \to q) \to (\Box p \to \Box q) \\ T &:= \Box p \to p \\ B &:= p \to \Box\Diamond p \end{aligned}$$

and rules: (MP), (Sub) i (RG).

Kripke frames for $\mathbf{KTB}$

Definition 1. *By a Kripke frame we mean a pair $\mathfrak{F} = \langle W, R \rangle$ where W -nonempty set and R relation on W.*

In the case of the logic $\mathbf{KTB}$, R is reflexive and symmetric.

Elements of W are called points and the relation R is an accessibility relation: xRy means: 'y is accessible from x'.

Valuation $\mathfrak{F}$ is a function $V : Var \rightarrow W$ and can be extended to homomorphism.

Then we define for each $x \in W$:

$$
\begin{array}{rlllll}
x \models p & \text{iff} & x \in V(p) & & & \\
x \models \alpha \wedge \beta & \text{iff} & x \models \alpha & \text{i} & x \models \beta & \\
x \models \alpha \vee \beta & \text{iff} & x \models \alpha & \text{or} & x \models \beta & \\
x \models \alpha \rightarrow \beta & \text{iff} & x \not\models \alpha & \text{or} & x \models \beta & \\
x \models \neg\alpha & \text{iff} & x \not\models \alpha & & & \\
x \models \Box\alpha & \text{iff} & \text{for any } y \in W & \text{if} & xRy & \text{then } y \models \alpha
\end{array}
$$

A formula α is a tautology of the logic $\mathbf{KTB}$, if it is true in every reflexive and symmetric Kripke model.

Extensions of $\mathbf{KTB}$

$\mathbf{T_n} = \mathbf{KTB} \oplus (4_n)$, where

$$(4_n) \qquad \Box^n p \to \Box^{n+1} p$$

$$(tran_n) \quad \forall_{x,y} \text{ (if } xR^{n+1}y \text{ then } xR^n y)$$

where the relation R^n is the n-step accessibility relation defined below:

$$xR^0 y \quad \text{iff} \quad x = y$$
$$xR^{n+1} y \quad \text{iff} \quad \exists_z \, (xR^n z \ \wedge \ zRy)$$

$$\mathbf{KTB} \subset \ldots \subset \mathbf{T_{n+1}} \subset \mathbf{T_n} \subset \ldots \subset \mathbf{T_2} \subset \mathbf{T_1} = \mathbf{S5}.$$

Definition 2. *A logic $\mathbf{L}$ is Kripke complete, if there is a class $\mathcal{C}$ of Kripke frames, such that:*

1. *for every formula $\psi \in \mathbf{L}$ and any frame $\mathfrak{F} \in \mathcal{C}$ we have $\mathfrak{F} \models \psi$.*

2. *for every formula $\psi \notin \mathbf{L}$, there is a Kripke frame $\mathfrak{G} \in \mathcal{C}$ such that $\mathfrak{G} \not\models \psi$.*

Fact 3. *The logics $\mathbf{T_n}$ are Kripke complete.*

Problem

Miyazaki in [1] defined one Kripke incomplete logic in $NEXT(\mathbf{T_2})$ and a continuum of Kripke incomplete logics in $NEXT(\mathbf{T_5})$.

Kostrzycka in [2] defined a continuum Kripke incomplete logics in $NEXT(\mathbf{T_2})$.

[1] Y. Miyazaki, Kripke incomplete logics containing KTB, Studia Logica **85**, (2007), 311-326.

[2] Kostrzycka Z, On non compact logics in $NEXT(\mathbf{KTB})$. Mathematical Logic Quarterly **54**, no. 6, (2008), 617-624.

Question: Is there a $\mathbf{KTB}$ - logic which is Kripke incomplete and finitely axiomatizable?

The aim

To define a logic L_* and a formula ψ such that $\psi \notin L_*$ and
for any Kripke frame $\mathfrak{F}$ the following implication holds:

if $\mathfrak{F} \models L_*$, then $\mathfrak{F} \models \psi$.

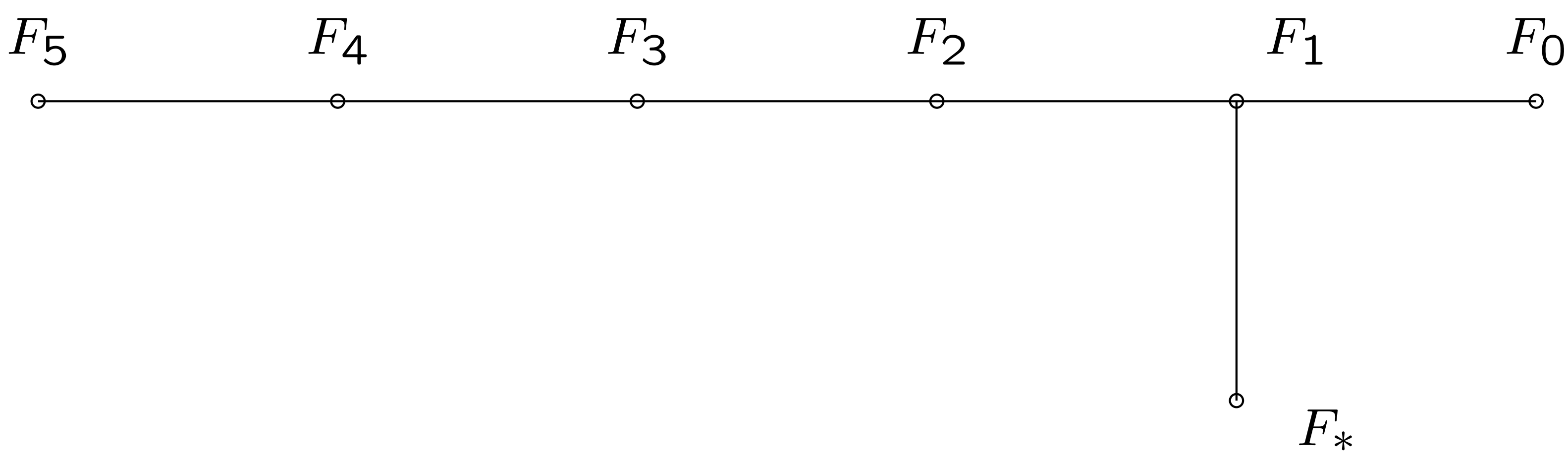
Axioms for L_*

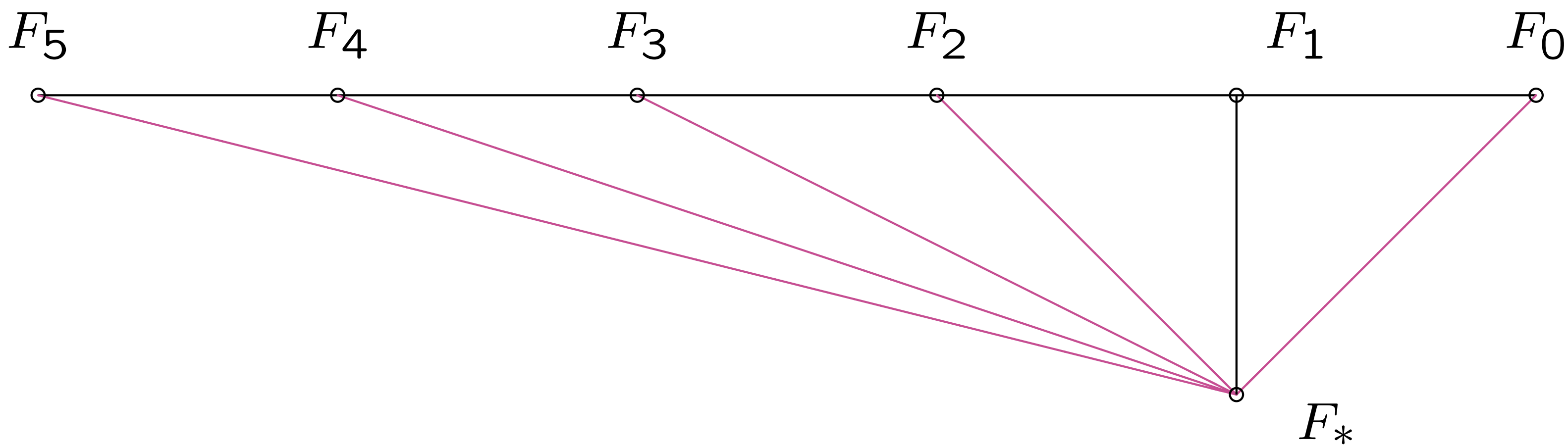
Exclusive formulas:

$$
\begin{aligned}
F_* &:= p_* \wedge \neg p_0 \wedge \neg p_1 \wedge \neg p_2 \wedge \neg p_3 \wedge \neg p_4 \wedge \neg p_5,\\
F_0 &:= \neg p_* \wedge p_0 \wedge \neg p_1 \wedge \neg p_2 \wedge \neg p_3 \wedge \neg p_4 \wedge \neg p_5,\\
F_1 &:= \neg p_* \wedge \neg p_0 \wedge p_1 \wedge \neg p_2 \wedge \neg p_3 \wedge \neg p_4 \wedge \neg p_5,\\
F_2 &:= \neg p_* \wedge \neg p_0 \wedge \neg p_1 \wedge p_2 \wedge \neg p_3 \wedge \neg p_4 \wedge \neg p_5,\\
F_3 &:= \neg p_* \wedge \neg p_0 \wedge \neg p_1 \wedge \neg p_2 \wedge p_3 \wedge \neg p_4 \wedge \neg p_5,\\
F_4 &:= \neg p_* \wedge \neg p_0 \wedge \neg p_1 \wedge \neg p_2 \wedge \neg p_3 \wedge p_4 \wedge \neg p_5,\\
F_5 &:= \neg p_* \wedge \neg p_0 \wedge \neg p_1 \wedge \neg p_2 \wedge \neg p_3 \wedge \neg p_4 \wedge p_5,
\end{aligned}
$$

$$
\begin{aligned}
Q \;\; := \;\; & \{F_1 \wedge \Diamond F_* \wedge \Diamond(F_0 \wedge \neg\Diamond F_2 \wedge \neg\Diamond F_3 \wedge \neg\Diamond F_4) \wedge \\
& \wedge\Diamond(F_2 \wedge \Diamond(F_3 \wedge \Diamond(F_4 \wedge \Diamond F_5)) \wedge \neg\Diamond F_4 \wedge \neg\Diamond F_5) \wedge \neg\Diamond F_3\} \\
& \to \Diamond(F_* \wedge \Diamond F_0 \wedge \Diamond F_2 \wedge \Diamond F_3 \wedge \Diamond F_4 \wedge \Diamond F_5).
\end{aligned}
$$

The role of Q:

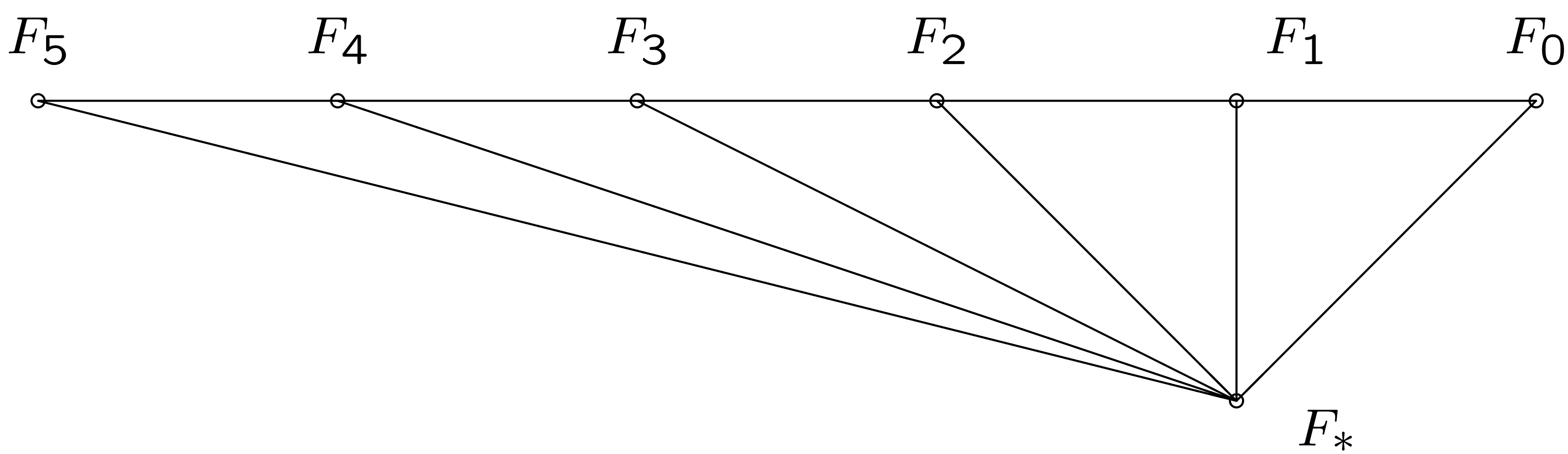


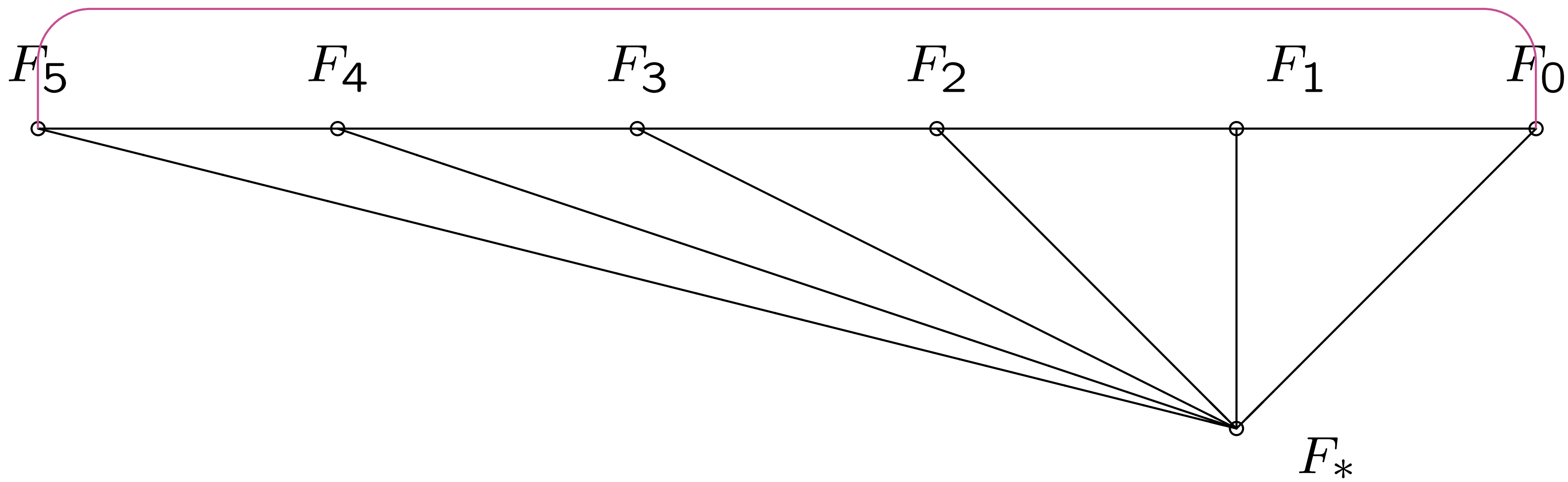


F_5
F_4
F_3
F_2
F_1
F_0
F_*

$$
\begin{aligned}
G \quad := \quad & \{F_5 \wedge \Diamond[F_4 \wedge \Diamond(F_3 \wedge \Diamond(F_2 \wedge \Diamond(F_1 \wedge \Diamond F_0)))] \wedge \Diamond F_* \\
& \wedge \left(\bigwedge_{i=0}^{5} F_i \rightarrow \Box p \right) \wedge \Box^2 \left((p \wedge F_*) \rightarrow \bigwedge_{i=0}^{5} \Diamond F_i \right) \wedge \\
& \wedge \Box^2 \left(F_* \vee \bigvee_{i=0}^{5} F_i \right) \wedge \Box^2[\Box(F_5 \vee (F_* \wedge p)) \rightarrow \Diamond(F_5 \wedge \Diamond F_4)] \wedge \\
& \wedge \bigwedge_{i=0}^{4} \Box^2[\Box(F_i \vee (F_* \wedge p)) \rightarrow \Diamond(F_i \wedge \Diamond F_{i+1})]\} \rightarrow \Diamond F_0
\end{aligned}
$$

The role of G:





F_5
F_4
F_3
F_2
F_1
F_0
F_*

$$
\begin{aligned}
P \;&:=\; \{r \wedge \bigwedge_{i=1}^{3} (A_i \wedge B_i \wedge C_i)\} \to \Diamond^2\{r \wedge \Box(r \to (q_1 \vee q_2 \vee q_3))\}, \\
&\quad \text{where} \\
A_i \;&:=\; \Box^2(q_i \to r), \quad B_i := \Box^2(r \to \Diamond q_i), \quad \text{for} \quad i = 1, 2, 3 \\
C_1 \;&:=\; \Box^2\neg(q_2 \wedge q_3), \quad C_2 := \Box^2\neg(q_1 \wedge q_3), \quad C_3 := \Box^2\neg(q_1 \wedge q_2).
\end{aligned}
$$

Definition 4. $L_* := \mathbf{T_2} \oplus G \oplus Q \oplus P$.

Formuła ψ

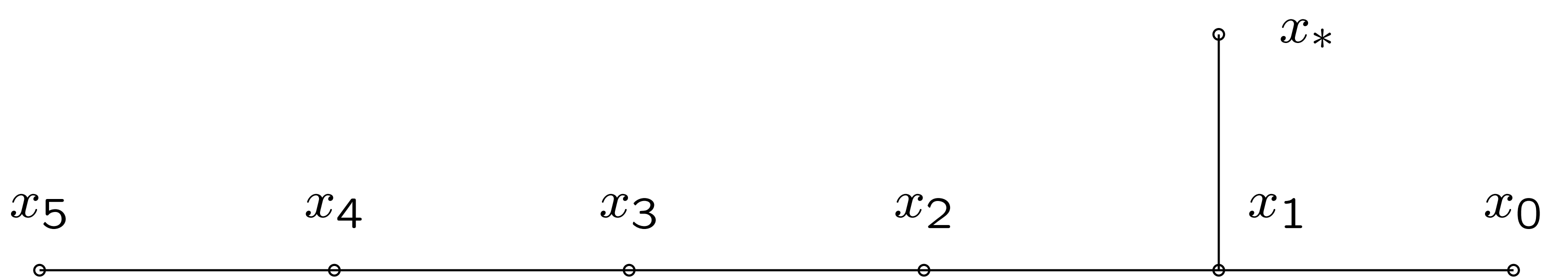
$$H_* := \neg s_0 \wedge \neg s_1 \wedge \neg s_2 \wedge \neg s_3 \wedge \neg s_4,$$
$$H_0 := \Box\neg s_0 \wedge \neg s_1 \wedge s_2 \wedge s_3 \wedge s_4,$$
$$H_1 := \neg s_0 \wedge \Box\neg s_1 \wedge \neg s_2 \wedge s_3 \wedge s_4,$$
$$H_2 := s_0 \wedge \neg s_1 \wedge \Box\neg s_2 \wedge \neg s_3 \wedge s_4,$$
$$H_3 := s_0 \wedge s_1 \wedge \neg s_2 \wedge \Box\neg s_3 \wedge \neg s_4,$$
$$H_4 := s_0 \wedge s_1 \wedge \neg s_2 \wedge \neg s_3 \wedge \Box\neg s_4,$$
$$H_5 := \neg s_0 \wedge s_1 \wedge \neg s_2 \wedge s_3 \wedge \neg s_4,$$

$$\psi := \neg\{H_5 \wedge \Diamond[H_4 \wedge \Diamond(H_3 \wedge \Diamond(H_2 \wedge \Diamond(H_1 \wedge \Diamond H_0 \wedge \Diamond H_*)))]\}.$$

Lemma 5. *For every Kripke frame* $\mathfrak{F}$ *it holds:*
if $\mathfrak{F} \models L_*$*, then* $\mathfrak{F} \models \psi$ *.*

Proof: Suppose that there is a Kripke frame $\mathfrak{F}$ such that $\mathfrak{F} \models L_*$ and $\mathfrak{F} \not\models \psi$.

Then the structure $\mathfrak{F}$ consists of at least seven different points $x_*, x_0, x_1, x_2, x_3, x_4, x_5$ such that: $x_1 R x_*$, and $x_i R x_j$ iff $|i - j| \leq 1$ for $i, j = 0, ..., 4$ i $x_4 R x_5$.

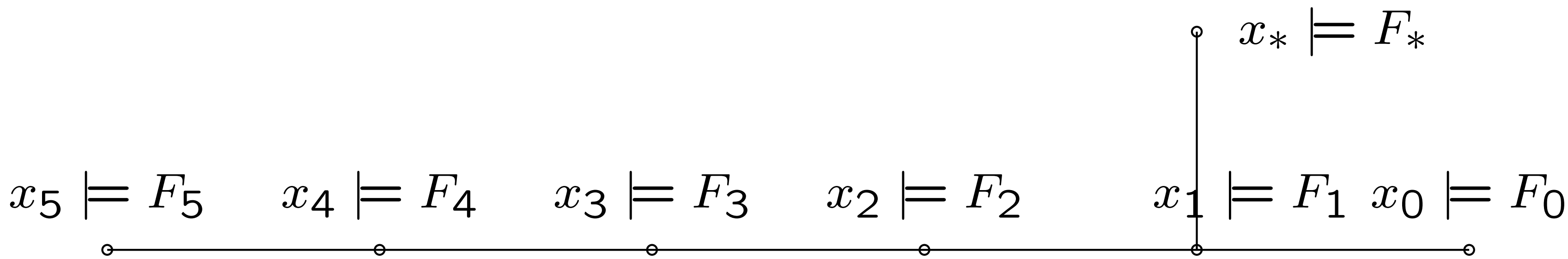


We define a valuation for the variables $p_0, ..., p_5, p_*$:

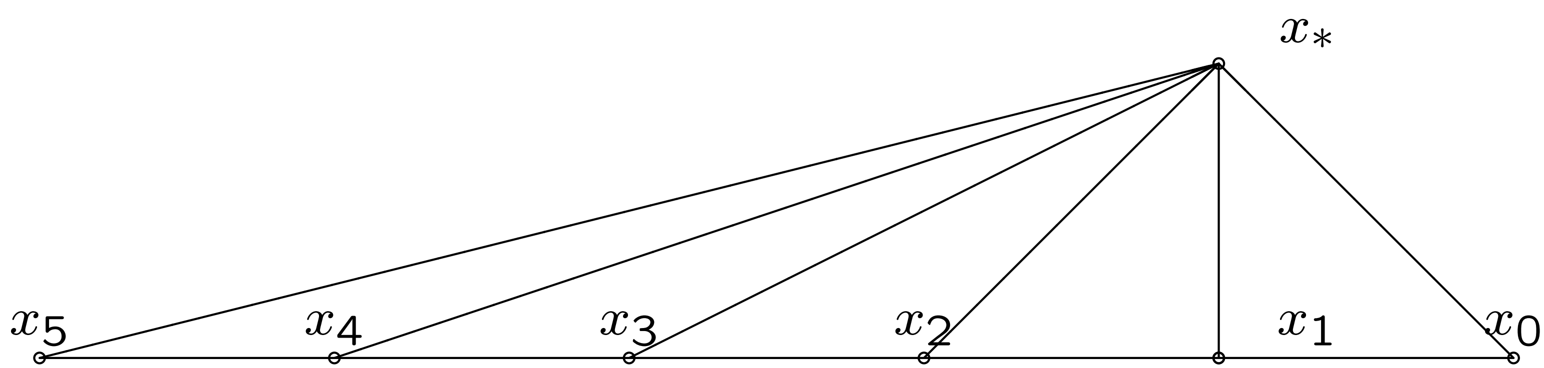
$$V(p_i) = \{x_i\} \quad \text{for} \quad i = 0, ..., 5, \quad \text{and} \quad V(p_*) = \{x_*\}.$$

That gives us:

$$V(F_i) = \{x_i\} \quad \text{for} \quad i = 0, ..., 5, \quad \text{and} \quad V(F_*) = \{x_*\}.$$



The formula Q has to be true under that valuation, hence it must hold: $x_* R x_j$, for $j = 0, 2, 3, 4, 5$.



Let us consider a new valuation defined on the obtained frame:

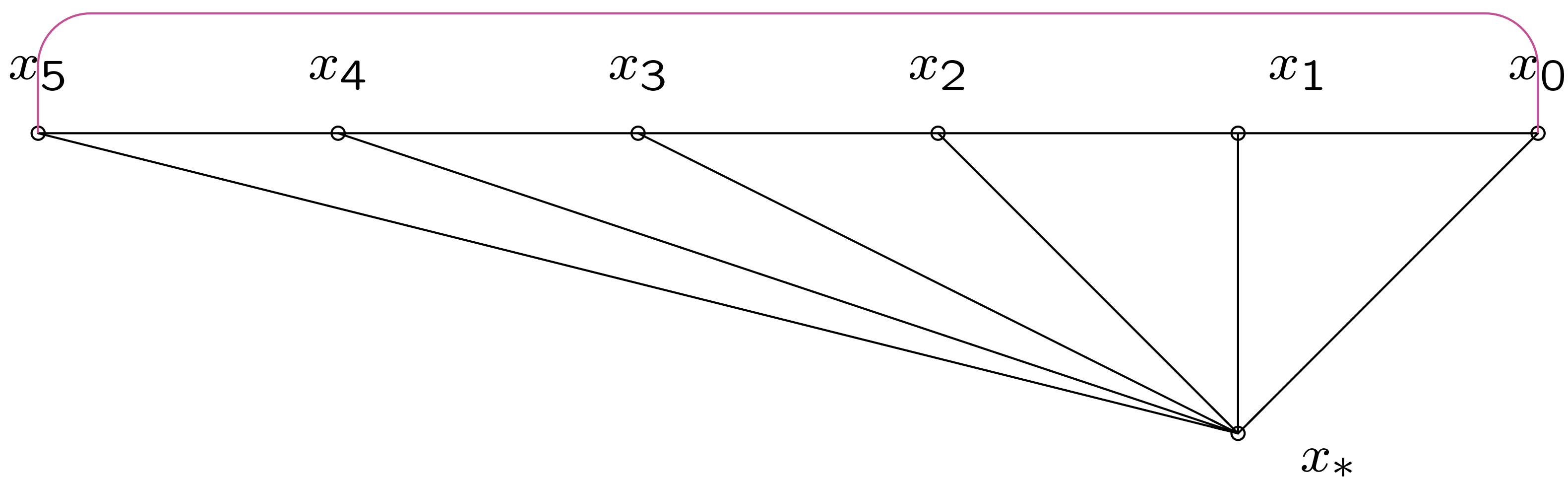
$$x_* \models p_*, \quad x_i \models p_i, \quad \text{for} \quad i = 0, 1, 2, 3, 4, 5$$

For any x such that x sees only the point x^* we define: $x \models p_*$ and $x \not\models p$.

In the other points we define: if xRy and $y \models p_i$, then $x \models p_k$ for $k \neq i$ and $i = 0, 1, ..., 5$ and $k = 1, ..., 4$. For such valuation we obtain:

$$x_* \models F_* \wedge p \quad \text{iff} \quad x = x_*$$
$$x \models F_0 \quad \text{iff} \quad x = x_0$$
$$x \models F_5 \quad \text{iff} \quad x = x_5$$

The antecedent of the formula G is true at x_5; the consequent of G has to be true at x_5 - hence $x_5 R x_0$. Then we obtain:



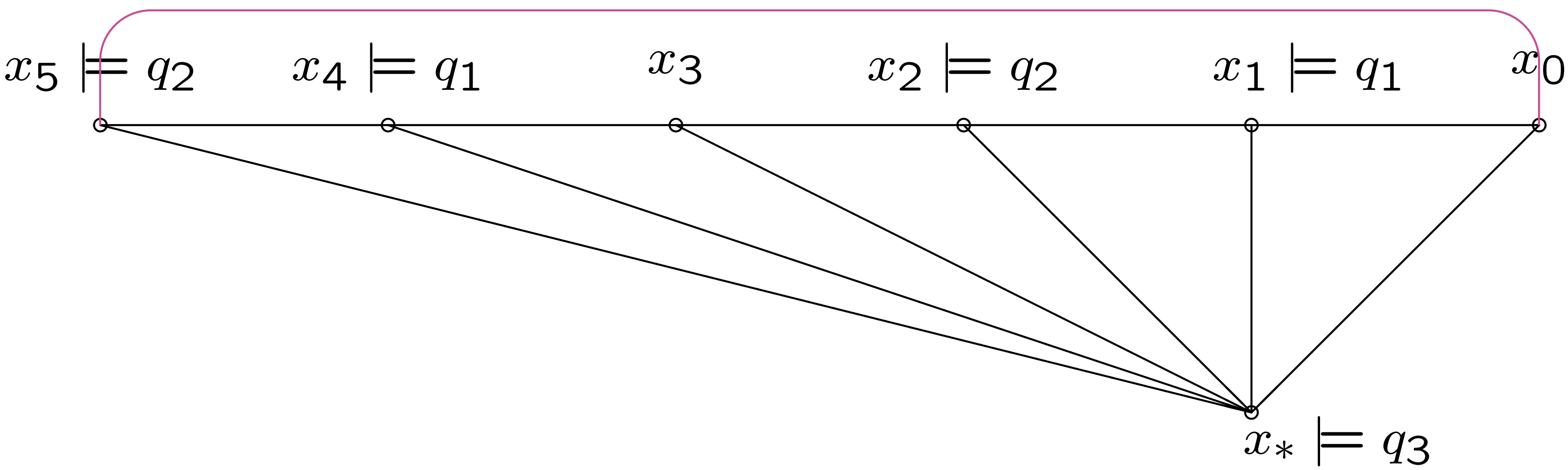
x_5
x_4
x_3
x_2
x_1
x_0
x_*

$$
\begin{aligned}
P \;&:=\; \{r \wedge \bigwedge_{i=1}^{3} (A_i \wedge B_i \wedge C_i)\} \rightarrow \Diamond^2\{r \wedge \Box(r \rightarrow (q_1 \vee q_2 \vee q_3))\}, \\
&\quad \text{where} \\
A_i \;&:=\; \Box^2(q_i \rightarrow r), \quad B_i := \Box^2(r \rightarrow \Diamond q_i), \quad \text{for} \quad i = 1, 2, 3 \\
C_1 \;&:=\; \Box^2\neg(q_2 \wedge q_3), \quad C_2 := \Box^2\neg(q_1 \wedge q_3), \quad C_3 := \Box^2\neg(q_1 \wedge q_2).
\end{aligned}
$$

Formula P is false with the following valuation:

$V_*(r) = \{x_*, x_0, ..., x_5\}$, $V_*(q_1) = \{x_1, x_4\}$, $V_*(q_2) = \{x_2, x_5\}$

$$V_*(q_3) = \{x_*\}.$$



We take x_3. It holds: $x_3 \models r$ and $x_3 \models A_i \wedge B_i \wedge C_i$ for $i = 1, 2, 3$. However $x_{3n} \not\models q_1 \vee q_2 \vee q_3$ for $n = 0, 1$, and then $x_3 \not\models \Diamond^2\{r \wedge \Box(r \to (q_1 \vee q_2 \vee q_3))\}$.

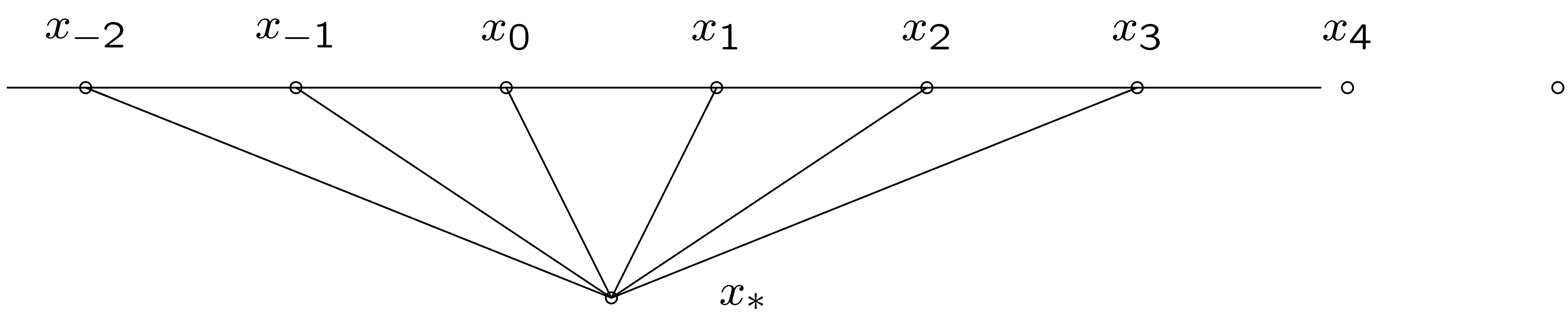
Hence: $x_3 \not\models P$.

Lemma 6. $\psi \notin L_*$.

Proof. We use a general frame. General frames are relational counterparts of modal algebras. Define:

$\mathfrak{G} = \langle W, R, T \rangle$ where:

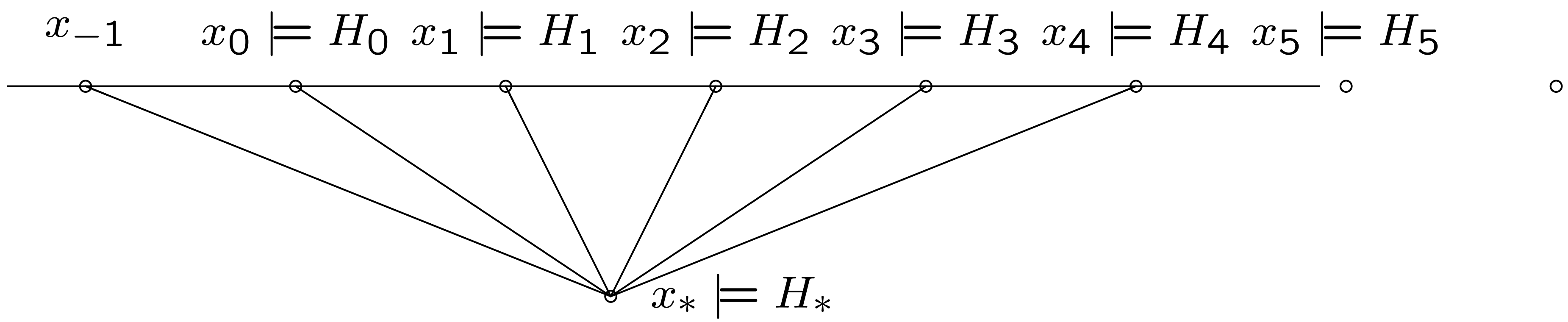
$$\begin{array}{rcl} W & := & \{x_*\} \cup \{x_i, i \in \mathbb{Z}\}, \\ R & := & \{(x_*, x_i \text{ for each } i \in \mathbb{Z}\} \cup \\ & & \cup\{(x_i, x_j) \text{ iff } |i-j| \ \leq 1; \text{ for any } i, j \ \in \ \mathbb{Z}\}, \\ T & := & \{X \subset W : X \text{ is finite or } W \setminus X \text{ is finite}\}. \end{array}$$



$\mathfrak{G} \models P, Q, G.$

Define a valuation:

$$V(s_0) = \{x_2, x_3, x_4\}, \ \ V(s_1) = \{x_3, x_4, x_5\}, \ \ V(s_2) = \{x_0, x_4, x_5\},$$
$$V(s_3) = \{x_0, x_1, x_5\}, \ \ V(s_4) = \{x_0, x_1, x_2\}.$$



Then for

$$\psi := \neg\{H_5 \wedge \Diamond[H_4 \wedge \Diamond(H_3 \wedge \Diamond(H_2 \wedge \Diamond(H_1 \wedge \Diamond H_0 \wedge \Diamond H_*)))]\}.$$

we obtain $\mathfrak{G} \not\models \psi$.

Theorem 7. *The logic* $L_* = \mathbf{T_2} \oplus G \oplus Q \oplus P$ *is Kripke incomplete.*

[3] Kostrzycka Z., *On a finitely axiomatizable Kripke incomplete logic containing KTB*, accepted at Journal of Logic and Computation.