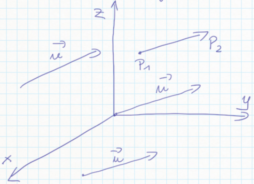
Kurs wyrównawczy - zajęcia zdalne z 9.12.2021

December 14, 2021

strona2.pdf

Wektory w $\mathbb{R}^3$



$P_1=(1,1,3)$, $P_2=(2,2,4)$

$\overrightarrow{P_1P_2}=[1,1,1]$

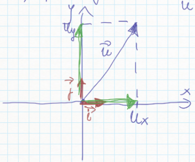
$\vec{u}=[1,1,1]$ wektor swobodny

$|\overrightarrow{P_1P_2}|=\sqrt{1^2+1^2+1^2}=\sqrt{3}$

strona3.pdf

Baza przestrzeni wektorowej:

1) płaszczyzna



$\vec{u} = [u_x, u_y]$

$\{\vec{i}, \vec{j}\}$ – baza na płaszcz.

Każdy wektor np. $\vec{u}$ można przedstawić jako ich kombinację liniową: $\underbrace{\alpha}_{u_x} \cdot \vec{i} + \underbrace{\beta}_{u_y} \cdot \vec{j}$

Mamy

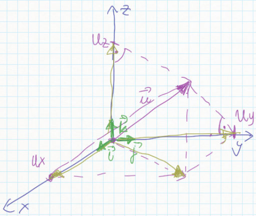
$$[u_x, u_y] = u_x \cdot \vec{i} + u_y \cdot \vec{j} =$$

$$= u_x \cdot [1, 0] + u_y \cdot [0, 1] =$$

$$= [u_x, 0] + [0, u_y] =$$

$$= [u_x, u_y]$$

w przestrzeni $\mathbb{R}^3$



$\vec{u} = [u_x, u_y, u_z]$

$\vec{u} = u_x \cdot \vec{i} + u_y \cdot \vec{j} + u_z \cdot \vec{k}$

$\vec{i} = [1,0,0]$

$\vec{j} = [0,1,0]$

$\vec{k} = [0,0,1]$

strona4.pdf

strona5.pdf

Zad. Dane są dwa wektory

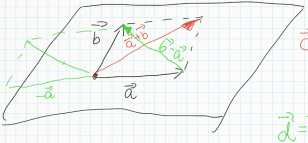
$\vec{a} = 3\vec{i} + 5\vec{j} - 4\vec{k}$, $\vec{b} = -\vec{i} + 5\vec{j} + 4\vec{k}$.

Na nich rozpięty równoległobok. Wyznacz długość przekątnych.

$\vec{a} = [3, 5, -4]$, $\vec{b} = [-1, 5, 4]$

$\vec{a} \nparallel \vec{b}$

$\vec{a} + \vec{b}$ = wektor będący przekątną równoległ. ....



$\vec{c} = \vec{a} + \vec{b} = [3, 5, -4] + [-1, 5, 4] =$

$= [2, 10, 0]$ $|\vec{c}| = \sqrt{4+100+0} = \sqrt{104}$

$\vec{d} = \vec{b} - \vec{a} = [-1, 5, 4] + [-3, -5, 4] =$

$= [-4, 0, 8]$ $|\vec{d}| = \sqrt{80}$

strona6.pdf

Iloczyn skalarny

Df. $\vec{a} \circ \vec{b} = |\vec{a}||\vec{b}| \cdot \cos \sphericalangle(\vec{a}, \vec{b})$

Jeżeli $\vec{a} \neq \vec{0}$ i $\vec{b} \neq \vec{0}$ i $\vec{a} \circ \vec{b} = 0 \Rightarrow \vec{a} \perp \vec{b}$

Inne własności:

1) $\vec{a} \circ \vec{b} = \vec{b} \circ \vec{a}$

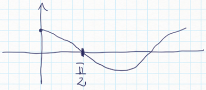
2) rozdz. dod. wzg. il. skal.

$\vec{a} \circ (\vec{b} + \vec{c}) = \vec{a} \circ \vec{b} + \vec{a} \circ \vec{c}$

3) łączność mnożenia skalarnego i przez liczbę $\lambda \in \mathbb{R}$

$(\lambda \vec{a}) \circ \vec{b} = \lambda (\vec{a} \circ \vec{b})$

4) $\vec{a} \circ \vec{a} = |\vec{a}|^2 \cdot \underbrace{\cos 0}_{1} \longrightarrow \vec{a} \circ \vec{a} = |\vec{a}|^2$



strona7.pdf

Iloczyn skalarny dwóch wektorów:

$\vec{a} = [a_x, a_y, a_z]$, $\vec{b} = [b_x, b_y, b_z]$

$$\boxed{\vec{a} \circ \vec{b} = a_x \cdot b_x + a_y \cdot b_y + a_z \cdot b_z}$$

$\vec{a} = a_x \vec{i} + a_y \vec{j} + a_z \vec{k}$, $\vec{b} = b_x \vec{i} + b_y \vec{j} + b_z \vec{k}$

$$\vec{a} \circ \vec{b} = (a_x \vec{i} + a_y \vec{j} + a_z \vec{k}) \circ (b_x \vec{i} + b_y \vec{j} + b_z \vec{k}) =$$

$$= a_x b_x \vec{i} \circ \vec{i} + \underbrace{a_x b_y \vec{i} \circ \vec{j}}_{=0} + \underbrace{a_x \cdot b_z \cdot \vec{i} \circ \vec{k}}_{=0} +$$

$$+ \underbrace{a_y b_x \vec{j} \circ \vec{i}}_{0} + a_y b_y \vec{j} \circ \vec{j} + \underbrace{a_y b_z \vec{j} \circ \vec{k}}_{=0} +$$

$$+ \underbrace{a_z b_x \vec{k} \circ \vec{i}}_{0} + \underbrace{a_z \cdot b_y \vec{k} \circ \vec{j}}_{0} + a_z \cdot b_z \vec{k} \circ \vec{k} = a_x b_x + a_y b_y + a_z b_z$$

$\vec{a} \perp \vec{b}$

$\vec{a} \circ \vec{b} = |\vec{a}| \cdot |\vec{b}| \cdot \underbrace{\cos 90^\circ}_{0}$

$\vec{a} \circ \vec{b} = 0$

$\vec{i} \perp \vec{j}$ $\quad \vec{i} \perp \vec{k}$

$\vec{j} \perp \vec{k}$

$\vec{i} \circ \vec{j} = 0$

$\vec{i} \circ \vec{k} = 0$

$\vec{j} \circ \vec{k} = 0$

$\vec{i} \circ \vec{i} = |\vec{i}|^2 = 1$

$\vec{j} \circ \vec{j} = 1$

$\vec{k} \circ \vec{k} = 1$

strona8.pdf

Zad. Wyznacz kąt pomiędzy wektorami:

$$\vec{a} = [1, 0, 3]$$
$$\vec{b} = [0, 5, -1]$$

odp.
$$\sphericalangle(\vec{a}, \vec{b}) = \arccos\left(-\frac{3}{\sqrt{260}}\right)$$

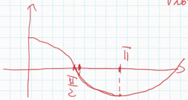
Ponieważ $\vec{a} \circ \vec{b} = |\vec{a}| \cdot |\vec{b}| \cos\sphericalangle(\vec{a}, \vec{b})$

$$\Downarrow$$

$$\cos\sphericalangle(\vec{a}, \vec{b}) = \frac{\vec{a} \circ \vec{b}}{|\vec{a}| \cdot |\vec{b}|}$$

$$\cos\sphericalangle(\vec{a}, \vec{b}) = \frac{-3}{\sqrt{10} \cdot \sqrt{26}} = \frac{-3}{\sqrt{260}} \approx -\frac{3}{16}$$

$$\vec{a} \circ \vec{b} = 1 \cdot 0 + 0 \cdot 5 + 3 \cdot (-1) = -3$$
$$|\vec{a}| = \sqrt{1^2 + 0^2 + 3^2} = \sqrt{10}$$
$$|\vec{b}| = \sqrt{0^2 + 5^2 + (-1)^2} = \sqrt{26}$$



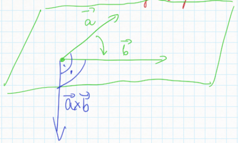
Iloczyn wektorowy

Df. Iloczynem wektorowym $\vec{a}\times\vec{b}$ wektorów $\vec{a}$ i $\vec{b}$ nazywamy wektor o następujących własnościach:

$$(\vec{a}\times\vec{b})\perp\vec{a} \quad \text{i} \quad (\vec{a}\times\vec{b})\perp\vec{b}$$

$$|\vec{a}\times\vec{b}| = |\vec{a}|\cdot|\vec{b}|\cdot\sin\sphericalangle(\vec{a},\vec{b})$$

i o zwrocie zgodnym z orientacją przestrzeni.



strona9.pdf

strona10.pdf

Własności

1) nieprzemienność: $\vec{b}\times\vec{a} = -(\vec{a}\times\vec{b})$

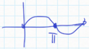
2) rozdzielność + wzg. $\times$

$$\vec{a}\times(\vec{b}+\vec{c}) = \vec{a}\times\vec{b} + \vec{a}\times\vec{c}$$

3) łączność $(\lambda\vec{a})\times\vec{b} = \lambda\cdot(\vec{a}\times\vec{b})$

4) Niech $\vec{a}\neq\vec{0}$ i $\vec{b}\neq\vec{0}$. Wtedy $\vec{a}\times\vec{b}=0 \iff \vec{a}\parallel\vec{b}$

$$|\vec{a}\times\vec{b}| = \underbrace{|\vec{a}|}_{\neq 0}\,\underbrace{|\vec{b}|}_{\neq 0}\cdot\sin\sphericalangle(\vec{a},\vec{b})$$



strona11.pdf

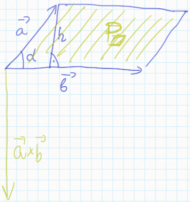
WZÓR : $\vec{a} = [a_x, a_y, a_z]$ $\quad \vec{b} = [b_x, b_y, b_z]$

$$\vec{a} \times \vec{b} = \begin{vmatrix} i & j & k \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = i \cdot (-1)^2 \begin{vmatrix} a_y & a_z \\ b_y & b_z \end{vmatrix} + j(-1)^3 \begin{vmatrix} a_x & a_z \\ b_x & b_z \end{vmatrix} + k(-1)^4 \begin{vmatrix} a_x & a_y \\ b_x & b_y \end{vmatrix}$$

$$= \left[a_y b_z - a_z b_y ,\ -(a_x b_z - b_x a_z) ,\ a_x b_y - a_y b_x \right]$$

strona12.pdf

WZÓR



$$P_{\square} = |\vec{a}\times\vec{b}|$$

$$\sin\alpha = \frac{h}{|\vec{a}|}$$

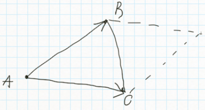
$$h = |\vec{a}|\cdot\sin\alpha$$

$$P_{\square} = |\vec{b}|\cdot|\vec{a}|\cdot\sin\alpha$$

$$P_{\square} = \underbrace{|\vec{a}|\cdot|\vec{b}|\cdot\sin\sphericalangle(\vec{a},\vec{b})}_{|\vec{a}\times\vec{b}|}$$

strona13.pdf

Zad. Obliczyć pole trójkąta o wierzchołkach
$A=(2,7,-1)$, $B=(0,3,5)$, $C=(-1,4,3)$.



$\overrightarrow{AB}=[-2,-4,6]$

$\overrightarrow{AC}=[-3,-3,4]$

$P_\Delta = \frac{1}{2}\,|\overrightarrow{AB}\times\overrightarrow{AC}|$

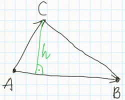
$P_\Delta = \frac{1}{2}\sqrt{2^2+(-10)^2+(-6)^2} = \frac{1}{2}\sqrt{140} = \sqrt{35}$

$$\overrightarrow{AB}\times\overrightarrow{AC} = \begin{vmatrix} i & j & k \\ -2 & -4 & 6 \\ -3 & -3 & 4 \end{vmatrix} = i(-1)^2\begin{vmatrix} -4 & 6 \\ -3 & 4 \end{vmatrix} + j(-1)^3\begin{vmatrix} -2 & 6 \\ -3 & 4 \end{vmatrix} + k(-1)^4\begin{vmatrix} -2 & -4 \\ -3 & -3 \end{vmatrix} =$$

$$= i(-16+18) - j(-8+18) + k(6-12) = 2i - 10j - 6k = [2,-10,-6]$$

strona14.pdf

Zad. Trójkąt ABC rozpięty jest na wektorach
$\vec{AB} = [1, 5, -3]$, $\vec{AC} = [-1, 0, 4]$.
Oblicz wysokość tego trójkąta opuszczoną z wierzchołka C.



$$P_\Delta = \frac{1}{2} \left| \vec{AB} \times \vec{AC} \right|$$

$$P_\Delta = \frac{1}{2} \cdot |\vec{AB}| \cdot h$$

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 5 & -3 \\ -1 & 0 & 4 \end{vmatrix} = i \cdot (-1)^{1+1} \cdot \begin{vmatrix} 5 & -3 \\ 0 & 4 \end{vmatrix} + j\,(-1)^{1+2} \begin{vmatrix} 1 & -3 \\ -1 & 4 \end{vmatrix} + k \cdot (-1)^4 \begin{vmatrix} 1 & 5 \\ -1 & 0 \end{vmatrix} =$$

$$= [20, -1, 5]$$

$$P_\Delta = \frac{1}{2}\left|\vec{AB}\times\vec{AC}\right| = \frac{1}{2}\cdot\sqrt{20^2+(-1)^2+5^2} = \frac{1}{2}\sqrt{400+26} = \frac{1}{2}\sqrt{426}$$

$$P_\Delta = \frac{1}{2}\left|\vec{AB}\right|\cdot h \quad \Rightarrow \quad h = \frac{2P_\Delta}{\left|\vec{AB}\right|}$$

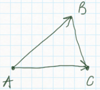
$$\vec{AB} = [1, 5, -3]$$

$$\left|\vec{AB}\right| = \sqrt{1^2+5^2+(-3)^2} = \sqrt{35} \qquad h = \frac{\sqrt{426}}{\sqrt{35}} = \sqrt{\frac{426}{35}}$$

strona15.pdf

Zad. Obliczyć pole trójkąta o wierzchołkach $A=(-1,2,-3)$, $B=(0,-1,1)$ i $C=(-1,1,0)$.

$$P_\Delta = \frac{1}{2}\left|\vec{a}\times\vec{b}\right|$$



$$\vec{AB} = [1,-3,4]$$

$$\vec{AC} = [0,-1,3]$$

$$\vec{AB}\times\vec{AC} = \begin{vmatrix} i & j & k \\ 1 & -3 & 4 \\ 0 & -1 & 3 \end{vmatrix} = i\begin{vmatrix} -3 & 4 \\ -1 & 3 \end{vmatrix} - j\begin{vmatrix} 1 & 4 \\ 0 & 3 \end{vmatrix} + k\begin{vmatrix} 1 & -3 \\ 0 & -1 \end{vmatrix} = [-5,-3,-1]$$

$$|\vec{AB}\times\vec{AC}| = \sqrt{25+9+1} = \sqrt{35} \qquad P_\Delta = \frac{1}{2}\sqrt{35}$$

strona16.pdf

strona17.pdf

Funkcje wykładnicze i logarytmiczna.

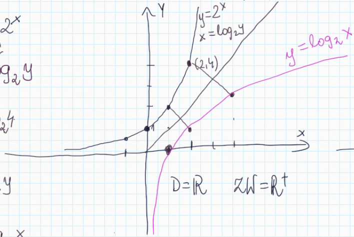
$y = 2^x$

$\Updownarrow$

$x = \log_2 y$

$\uparrow$

$2 = \log_2 4$



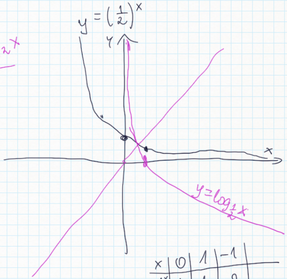
$D=\mathbb{R}$ $ZW=\mathbb{R}^+$

$x = \log_2 y$

$y \Longleftrightarrow x$

$y = \log_2 x$

$D = \mathbb{R}^+$, $ZW = \mathbb{R}$



x	0	1	−1
$\left(\frac{1}{2}\right)^x$	1	$\frac{1}{2}$	2

strona18.pdf

Zad. Narysuj wykres funkcji:

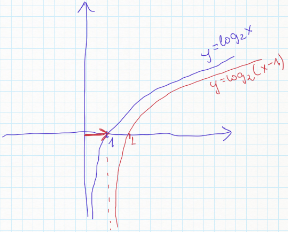
a) $y = \log_2(x-1)$

$D:\ x-1>0$

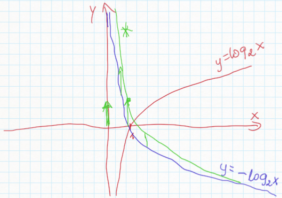
$x>1$

$y=f(x) \xrightarrow[\vec{w}=[a,b]]{} y=f(x-a)+b$

$\vec{w}=[1,0]$



b) $y = 1 - \log_2 x$



$y = \log_2 x$

$\downarrow$

$y_1 = -\log_2 x$

$\downarrow$

$y = -\log_2 x + 1$

strona19.pdf

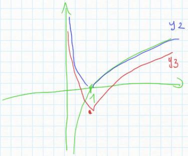
c) $y = |\log_2 x| - 1$

$y_1 = \log_2 x$

$y_2 = |\log_2 x|$

$\vec{w} = [0, -1] \downarrow$

$y_3 = |\log_2 x| - 1$

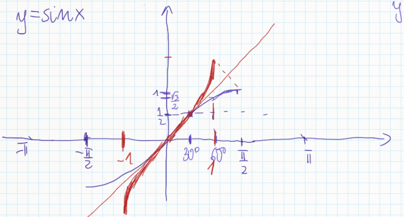


strona20.pdf

strona21.pdf

Funkcje cyklometryczne - odwrotne do trygonometrycznych (kwestia różnowartościowości!!!)

$y = \sin x$



$\arcsin\frac{\sqrt{2}}{2} = \frac{\pi}{4}$ bo $\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$

$y = \sin x \ \wedge \ x \in \langle -\frac{\pi}{2}, \frac{\pi}{2} \rangle$ - funkcja jest różnowart.

$\Updownarrow$

$x = \arcsin y$

kąt

liczba z $\langle -1, 1 \rangle$

$x \leftrightarrow y$

$y = \arcsin x$

$D = \langle -1, 1 \rangle \quad ZW = \langle -\frac{\pi}{2}, \frac{\pi}{2} \rangle$

strona22.pdf

$$\sin \sphericalangle(\vec{a}, \vec{b}) = \sqrt{\frac{23}{24}}$$

$$\sphericalangle(\vec{a}, \vec{b}) = \arcsin \sqrt{\frac{23}{24}}$$

$y = \cos x \qquad x \in \langle 0, \pi \rangle$

– różnowartościowa

$\iff \quad x = \arccos y$

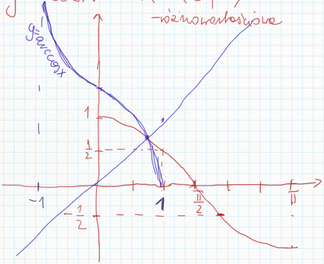
kąt

liczba z $\langle -1, 1 \rangle$

$x \leftrightarrow y$

$y = \arccos x$

$D = \langle -1, 1 \rangle \qquad ZW = \langle 0, \pi \rangle$



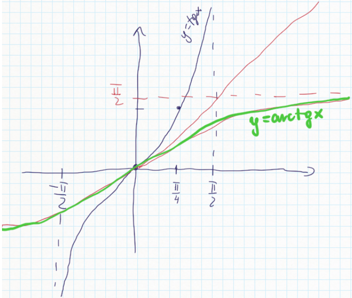
$\arccos 0 = \frac{\pi}{2}$ bo $\cos \frac{\pi}{2} = 0$

strona23.pdf

strona24.pdf

$y = tg\, x \quad x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \iff x = arctg\, y$

kąt   liczby



$arctg\, 0 = \boxed{0} \Rightarrow tg\, \boxed{0} = 0$

$arctg\, 1 = \boxed{\frac{\pi}{4}} \Rightarrow tg\, \boxed{\frac{\pi}{4}} = 1$

$arctg\, \sqrt{3} = \boxed{\frac{\pi}{3}} \Rightarrow tg\, \frac{\pi}{3} = \sqrt{3}$

$x \leftrightarrow y$

$y = arctg\, x$

$D = \mathbb{R}$ i $ZW = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$