Kurs wyrównawczy - zajęcia zdalne z 14.12.2021

December 14, 2021

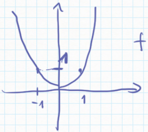
Funkcje różnowartościowe:

$$\forall_{x_1,x_2\in X}\ \left(x_1 \neq x_2 \Leftrightarrow f(x_1)\neq f(x_2)\right) \qquad (p\Leftrightarrow q)\Leftrightarrow(\sim p\Leftrightarrow \sim q)$$

$$\Updownarrow$$

$$x_1=x_2 \Leftrightarrow f(x_1)=f(x_2)$$

np. $f(x)=x^2$



$f(1)=f(-1)=1$ i $1\neq -1$

ta funkcja nie jest różnowartościowa

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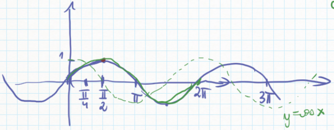
Wszystkie $y=ax^2+bx+c$ nie są różnowartościowe

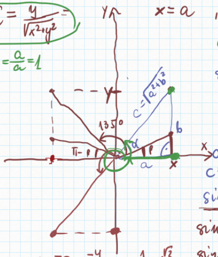
Inny przykład $y=\sin x$



$\alpha=x$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3}{4}\pi$	π	$\frac{5}{4}\pi$	$\frac{3}{2}\pi$	2π	$\frac{9}{4}\pi$
$\sin x$	0	$\frac{\sqrt{2}}{2}$	1	$\frac{\sqrt{2}}{2}$	0	$-\frac{\sqrt{2}}{2}$	-1	0	$\frac{\sqrt{2}}{2}$
cos	1								

$\sin\alpha = \frac{y}{\sqrt{x^2+y^2}}$

$\cos 0 = \frac{a}{a} = 1$



$x=a$, $y=b$

$\cos\alpha = \frac{a}{c}$

$\sin\alpha = \frac{b}{c}$

$\sin 0 = \frac{0}{c} = 0$

$\alpha = 45^0$

$c=\sqrt{a^2+a^2}=\sqrt{2}\cdot a$

$\sin 45^0 = \frac{a}{\sqrt{2}a} = \frac{\sqrt{2}}{2}$

$\sin 90^0 = \frac{b}{b} = 1$

$\sin\alpha = \sin(\pi-\alpha)$

$\sin 225^0 = \frac{-y}{\sqrt{2}y} = -\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

$\sin\frac{3}{2}\pi = \frac{-y}{y} = -1$

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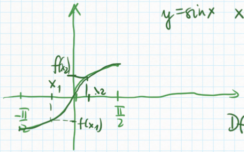
np. $\sin\left(\frac{\pi}{4}\right) = \sin\left(\frac{3}{4}\pi\right) = \frac{\sqrt{2}}{2}$

różne argum.

funkcja nie jest różnow.

$y = \sin x \quad x \in \langle -\frac{\pi}{2}, \frac{\pi}{2} \rangle \Rightarrow$ jest różnowartościowa



Tw. Funkcja jest różnow. w $\langle a, b \rangle$, gdy jest ściśle rosnąca lub ściśle malejąca..

Df. $f \nearrow \quad \forall_{x_1, x_2 \in \langle a, b \rangle} \quad x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$

Df. $f \searrow \quad x_1 < x_2 \quad >$

np. $(g\circ f)(x) = g(f(x))$ $\qquad X, Y, Z \subseteq \mathbb{R}$

$$(g\circ f\circ f)(x) = g(f(f(x))) = g(f(x^2)) = g(x^4) = x^4+5$$

$f(x) = x^2$

$g(x) = x+5$

$$\begin{tikzcd} x \arrow[r, "f"] \arrow[rr, bend right, "(g\circ f)(x)=x^2+5"'] & x^2 \arrow[r, "g"] & x^2+5 \end{tikzcd}$$

Jak wygląda złożenie w innej kolejności:

$$\begin{tikzcd} x \arrow[r, "g"] \arrow[rr, bend right, "(f\circ g)(x)=(x+5)^2"'] & x+5 \arrow[r, "f"] & (x+5)^2 \end{tikzcd}$$

Trzeba rozumieć zapis:

$y = (x+5)^2$

$y_1 = x+5$ ← funkcja wewn.

$y_2 = x^2$ ← funkcja zew.

$$g\circ f \neq f\circ g$$

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Lista 4.

Zad. 3. Z jakich funkcji elementarnych złożone są następujące funkcje:

a) $y = \sin^3 x = (\sin x)^3$

funkcja wewn. $f_1(x) = \sin x$

funkcja zew. $f_2(x) = \underline{x^3}$

$(f_2 \circ f_1)(x) = f_2(f_1(x)) = f_2(\sin x) = (\sin x)^3 = \sin^3 x$

b) $y = \log(\operatorname{tg} x)$

funkcja wew. $f_1(x) = \operatorname{tg} x$

zew. $f_2(x) = \log x$

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strona9.pdf

d) $y=\sqrt{x^2+5}$

$f_1(x)=x^2+5 \quad \leftarrow$ f. wew.

$f_2(x)=\sqrt{x} \quad \leftarrow$ f. zewn.

$f_2(f_1(x))=f_2(x^2+5)=\sqrt{x^2+5}$

e) $y=\sin^3(2x+1)=\left(\sin(2x+1)\right)^3$

f. wew. $\quad f_1(x)=2x+1$

f. środ. $\quad f_2(x)=\sin x$

f. zew. $\quad f_3(x)=x^3$

rozbić: $y = \sqrt[3]{\log_2(x+1) - 2}$

$(f_1 \circ f_2)(x) = \log_2 x + 1$

f. wew.

$f_1(x) = x+1$

$f_2(x) = \log_2 x$

$f_3(x) = x-2$

$f_4(x) = \sqrt[3]{x}$

$$f_4(f_3(f_2(f_1(x)))) = f_4(f_3(f_2(x+1))) = f_4(f_3(\log_2(x+1))) =$$

$$= f_4(\log_2(x+1) - 2) = \sqrt[3]{\log_2(x+1)-2}$$

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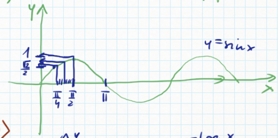
Zad. 2. Dana jest funkcja $g(x): (0, \pi) \to \mathbb{R}$ określona $g(x) = \log_2(\sin x)$. Wyznaczyć $g(B)$ dla $B = \langle \frac{\pi}{4}, \frac{\pi}{2} \rangle$.

$g_1(x) = \sin x$

$g_2(x) = \log_2 x$

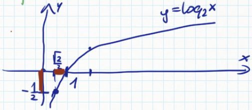
$$\begin{tikzcd} \langle \frac{\pi}{4}, \frac{\pi}{2} \rangle \arrow[r, "g_1"] \arrow[rr, bend right, "g_2 \circ g_1"'] & \langle \frac{\sqrt{2}}{2}, 1 \rangle \arrow[r, "g_2"] & \langle -\frac{1}{2}, 0 \rangle \end{tikzcd}$$



$\log_2 \frac{\sqrt{2}}{2} = \log_2 \frac{2^{\frac{1}{2}}}{2} = \log_2 2^{-\frac{1}{2}} = -\frac{1}{2}$

$\log_2 1 = 0$ bo $2^0 = 1$

$\log_2 \frac{\sqrt{2}}{2} = \log_2 2^{-\frac{1}{2}} = -\frac{1}{2}$ bo $2^{-\frac{1}{2}} = 2^{-\frac{1}{2}}$

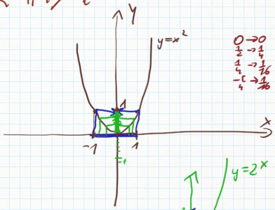


Wyznacz zbiór $f(A)$ gdzie $A=\langle -1,1\rangle$ i funkcja $y=2^{x^2}$.

wew. $\rightarrow$ $f_1(x)=x^2$

zew. $\rightarrow$ $f_2(x)=2^x$

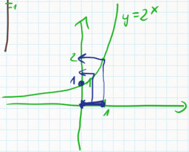
$0\to 0$, $\frac{1}{2}\to\frac{1}{4}$, $\frac{1}{4}\to\frac{1}{16}$, $-\frac{1}{4}\to\frac{1}{16}$



$$\langle -1,1\rangle \xrightarrow{f_1} \langle 0,1\rangle \xrightarrow{f_2} \langle 1,2\rangle$$

$f_1(\langle -1,1\rangle)$

odp. $f(A)=\langle 1,2\rangle$



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Zad. 1 Wyznaczyć dziedzinę funkcji:

a) $f(x) = \frac{1}{\sin 2x}$

$D: \sin 2x \neq 0$

$f_1(x) = 2x$

$f_2(x) = \sin x$

$f_3(x) = \frac{1}{x}$

Rozwiążemy: $\sin 2x = 0$

$t = 2x$

$\sin t = 0 \iff t = k\pi \quad k \in \mathbb{Z}$

$x^4 + 2x^2 + 1 = 0$

$x^2 = t$

$t^2 + 2t + 1 = 0$

$2x = k\pi \quad /:2$

$x = \frac{1}{2} k\pi$

Odp. $D: x \neq \frac{k\pi}{2}, \ k \in \mathbb{Z}$

b) $g(x)=\sqrt{\sin x-\frac{1}{2}}$

$f_1(x)=\sin x$

$f_2(x)=x-\frac{1}{2}$

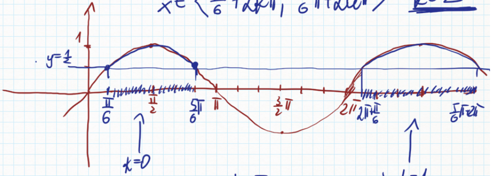
$f_3(x)=\sqrt{x}$

D: $\sin x-\frac{1}{2}\geq 0$

$\sin x\geq\frac{1}{2}$

całk.

$x\in\left\langle \frac{\pi}{6}+2k\pi, \frac{5}{6}\pi+2k\pi\right\rangle$ $\underline{k\in\mathbb{Z}}$



$x\in\bigcup_{k\in\mathbb{Z}}\left\langle\frac{\pi}{6}+2k\pi, \frac{5}{6}\pi+2k\pi\right\rangle$

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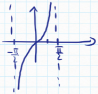
c) $h(x) = \log(\sqrt{3} - \operatorname{tg} x)$



$\operatorname{tg} x = \frac{\sin x}{\cos x}$

$D: x \neq \frac{\pi}{2} + k\pi, \ k \in Z$

$f_1(x) = \operatorname{tg} x \quad \leftarrow$

$x \neq \frac{\pi}{2} + k\pi$

$f_2(x) = \sqrt{3} - x$

$f_3(x) = \log x$

$x > 0$

$\sqrt{3} - \operatorname{tg} x > 0$

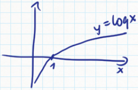
$\sqrt{3} > \operatorname{tg} x$

$\operatorname{tg} x < \sqrt{3}$

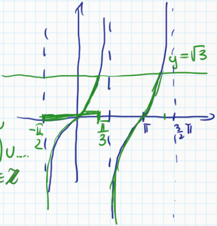
$x \in \left(-\frac{\pi}{2}, \frac{\pi}{3}\right) \cup$

$\cup \left(\frac{\pi}{2}, \frac{4}{3}\pi\right) \cup \ldots$

$x \in \left(-\frac{\pi}{2} + k\pi, \frac{\pi}{3} + k\pi\right), \ k \in \mathbb{Z}$



$\operatorname{tg} 60^\circ = \sqrt{3}$



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e) $k(x) = \log_2(1-2\cos x)$

$f_1(x) = \cos x \quad \leftarrow \mathbb{R}$

$f_2(x) = 1-2x \quad \leftarrow \mathbb{R}$

$f_3(x) = \log_2 x \quad x \in \mathbb{R}^+$

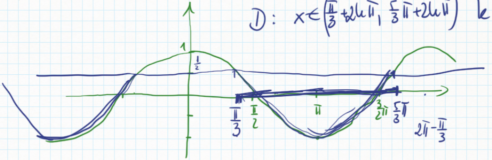
D: $1-2\cos x > 0$

$1 > 2\cos x$

$2\cos x < 1 \quad |:2$

$\cos x < \frac{1}{2}$

D: $x \in \left(\frac{\pi}{3}+2k\pi, \frac{5}{3}\pi+2k\pi\right) \quad k \in \mathbb{Z}$



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Dziedzina funkcji

$y = x \quad D = \mathbb{R}$

$y = \frac{1}{x} \quad D = \mathbb{R} \setminus \{0\}$

$y = \sqrt{x} \quad D = \langle 0, +\infty)$

$x \geq 0$

$y = \log_2 x \quad D = \mathbb{R}^+$

$x > 0$

$y = 2^x \quad D = \mathbb{R}$

$\left.\begin{array}{l} y = \sin x \\ y = \cos x \end{array}\right\} D = \mathbb{R}$

$ZW = \langle -1, 1 \rangle$

$y = \operatorname{tg} x \quad x \neq \frac{\pi}{2} + k\pi$

strona18.pdf

Iloczyn mieszany

$$\vec{a}\vec{b}\vec{c} = (\vec{a} \times \vec{b}) \circ \vec{c}$$

$$\vec{a} = [a_x, a_y, a_z]$$

$$\vec{b} = [b_x, b_y, b_z]$$

$$\vec{c} = [c_x, c_y, c_z]$$

$$\vec{a}\vec{b}\vec{c} = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix} \begin{matrix} a_x & a_y \\ b_x & b_y \\ c_x & c_y \end{matrix} = a_x b_y c_z + a_y b_z c_x + a_z b_x c_y - a_z b_y c_x - a_x b_z c_y - a_y b_x c_z$$

spr: $$\vec{a} \times \vec{b} = \begin{vmatrix} i & j & k \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = [a_y b_z - a_z b_y, -(a_x b_z - b_x a_z), a_x b_y - b_x a_y]$$

$$(\vec{a} \times \vec{b}) \circ \vec{c} = (a_y b_z - a_z b_y) \cdot c_x + (b_x a_z - a_x b_z) \cdot c_y + (a_x b_y - b_x a_y) \cdot c_z$$

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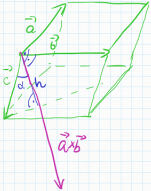


$h \parallel \vec{a}\times\vec{b}$

WZÓR $\vec{w}\circ\vec{v} = |\vec{w}|\cdot|\vec{v}|\cdot\cos\sphericalangle(\vec{w},\vec{v})$

$V = P_p \cdot h$

$P_p = |\vec{a}\times\vec{b}|$

$\frac{h}{|\vec{c}|} = \cos\alpha \Rightarrow h = |\vec{c}|\cdot\cos\alpha$

$\alpha = \sphericalangle(\vec{a}\times\vec{b}, \vec{c})$

$V = \underbrace{|\vec{a}\times\vec{b}|}_{u}\cdot\underbrace{|\vec{c}|}_{v}\cdot\cos\sphericalangle(\vec{a}\times\vec{b},\vec{c})$

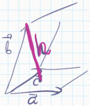
$*$ ← kąt ostry

$V = |(\vec{a}\times\vec{b})\circ\vec{c}|$

Zad. Obliczyć objętość równoległościanu zbudowanego na wektorach $\vec{a}=[0,-2,5]$, $\vec{b}=[1,3-2]$, $\vec{c}=[4,-1,3]$.

Obliczyć wysokość opuszczoną na podstawę rozpiętą na wektorach $\vec{a}$ i $\vec{c}$.

$$\vec{a}\vec{b}\vec{c} = \left|\begin{array}{ccc} 0 & -2 & 5 \\ 1 & 3 & -2 \\ 4 & -1 & 3 \end{array}\right|\begin{array}{cc} 0 & -2 \\ 1 & 3 \\ 4 & -1 \end{array} = 0+16-5-60-0+6 = 17-60=-43$$



$$h=\frac{43}{\sqrt{465}}$$

$$V=|\vec{a}\vec{b}\vec{c}|=43$$

$$V=P_p\cdot h \Rightarrow h=\frac{V}{P_p}$$

$$\vec{a}\times\vec{c}=\left|\begin{array}{ccc} i & j & k \\ 0 & -2 & 5 \\ 4 & -1 & 3 \end{array}\right|=[-1,20,8] \qquad P_p=|\vec{a}\times\vec{c}|=\sqrt{(-1)^2+20^2+8^2}=\sqrt{465}$$

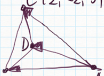
strona20.pdf

strona21.pdf

Zadanie z listy.

$V_{cz} = \frac{1}{6} V_r \qquad V_{cz} = 5$

$C(2,-2,3) \qquad D=(0,y,0)$



$B(3,-1,1) \qquad A(2,0,-1)$

$\vec{AB} = [1,-1,2]$

$\vec{AC} = [0,-2,4]$

$\vec{AD} = [-2,y,1]$

$$\frac{|(\vec{AB}\,\vec{AC}\,\vec{AD})|}{6} = 5 \quad |\cdot 6$$

$$|(\vec{AB}\,\vec{AC}\,\vec{AD})| = 30$$

$$\left|\begin{array}{ccc|cc} 1 & -1 & 2 & 1 & -1 \\ 0 & -2 & 4 & 0 & -2 \\ -2 & y & 1 & -2 & y \end{array}\right| =$$

$$= -2 + \cancel{8} + 0 - \cancel{8} - 4y - 0 = -2-4y$$

$V = |-2-4y|$

Mamy

$|-2-4y| = 30$

$1^\circ \quad -2-4y \geq 0$

$-2-4y = 30$

$-4y = 32 \quad |:-4$

$y=-8$

$2^\circ \quad -2-4y < 0$

$2+4y = 30$

$4y = 28$

$y = 7$

odp. $y=7 \vee y=-8$