On formulas with one variable in some fragment of Grzegorczyk's logic

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Abstract

In this paper we examine normal extensions of Grzegorczyk's logic over language with one propositional variable and signs of $\{\rightarrow, \Box\}$ only.

1 Grzegorczyk's logic and its normal extensions

Syntactically, Grzegorczyk's logic **Grz** is characterized as a normal extension of **S4** modal calculus by the axiom

$$(grz)\ \ \Box(\Box(p \rightarrow \Box p) \rightarrow p) \rightarrow p$$

The set of rules consists of modus ponens, substitution and necessitation.
Semantically, **Grz** logic is characterized by the class of finite reflexive and transitive trees. Recall, that by a tree we mean a rooted frame $\mathfrak{F} =< W, R >$ such that for every point $x \in W$, the set $x \downarrow$ is finite and linearly ordered by R.
In this section we examine normal extensions of Grzegorczyk's logic obtained by adding to the set of axioms new formulas. The axiomatic extensions are uniformly connected with a depth of a tree.

Definition 1. *A frame $\mathfrak{F}$ is of depth $n < \omega$ if there is a chain of n points in $\mathfrak{F}$ and no chain of more than n points exists in $\mathfrak{F}$.*

For $n > 0$, let J_n be an axiom saying that any strictly ascending partial-ordered sequence of points is of length n at most, i.e., that there exist no points $x_1, x_2, ..., x_n$ such that x_{n+1} is accessible from x_i for $i = 1, 2, ..., n$.
The formulas J_n are well known (see [1] pp.42) and are defined inductively as follows:

Definition 2.

$$\begin{aligned} J_1 &= \Diamond\Box p_1 \rightarrow p_1, \\ J_{n+1} &= \Diamond(\Box p_{n+1} \wedge \sim J_n) \rightarrow p_{n+1}. \end{aligned}$$

We will consider the logics $\mathbf{Grz}^{\leq n} = \mathbf{Grz} \oplus J_n$. They contain the logic **Grz** and the following inclusions hold:

$$\mathbf{Grz} \subset ... \subset \mathbf{Grz}^{\leq n} \subset \mathbf{Grz}^{\leq n-1} \subset ... \subset \mathbf{Grz}^{\leq 2} \subset \mathbf{Grz}^{\leq 1} \tag{1}$$

2 Implicational-necessitional reducts of extensions of Grzegorczyk's logic

The main goal of this paper is to investigate the implicational-necessitional reduct of Grzegorczyk's logic with one variable. The language $\mathcal{F}^{\{\rightarrow,\Box\}}$ consisted of sings of implication and necessity and one propositional variable p only is defined as follow:

Definition 3.

$$\begin{array}{l} p \in \mathcal{F}^{\{\rightarrow,\Box\}} \\ \alpha \rightarrow \beta \in \mathcal{F}^{\{\rightarrow,\Box\}} \quad \textit{iff} \;\; \alpha \in \mathcal{F}^{\{\rightarrow,\Box\}} \;\; \textit{and} \;\; \beta \in \mathcal{F}^{\{\rightarrow,\Box\}} \\ \Box\alpha \in \mathcal{F}^{\{\rightarrow,\Box\}} \quad \textit{iff} \;\; \alpha \in \mathcal{F}^{\{\rightarrow,\Box\}}. \end{array}$$

From now on, the whole investigation will concern implicational-necessitional reducts of logics **Grz** and $\mathbf{Grz}^{\leq n}$. We do not introduce new symbols for them. The simplest manner to characterize the logics $\mathbf{Grz}^{\leq n}$ is examining the appropriate Tarski-Lindenbaum algebras $\mathbf{Grz}^{\leq n}/_{\equiv}$. Let us introduce an equivalence relation on algebras $\mathbf{Grz}^{\leq n}$:

Definition 4. *$\alpha \equiv \beta$ iff $\alpha \rightarrow \beta \in \mathbf{Grz}^{\leq n}$ and $\beta \rightarrow \alpha \in \mathbf{Grz}^{\leq n}$ for $n = 1, 2, ..., n$.*

Lemma 5. *For any algebra $\mathbf{Grz}^{\leq n}/_{\equiv}$ the following orders hold:*

$$[\Box p]_{\equiv} \leq [\alpha]_{\equiv} \;\; \textit{for any} \;\; \alpha \in \mathcal{F}^{\{\rightarrow,\Box\}}, \tag{2}$$

$$[\alpha]_{\equiv} \leq [p \rightarrow p]_{\equiv} \;\; \textit{for any} \; \alpha \in \mathcal{F}^{\{\rightarrow,\Box\}}, \tag{3}$$

where $\leq$ is defined in the conventional way.

Proof. The proof of (3) is obvious. (2) follows from reflexivity of appropriate tree and restriction to the formulas from $\mathcal{F}^{\{\rightarrow,\Box\}}$. If the formula $\Box p$ is true at some point x of some tree it involves being true for p at every point $x_i \in x \uparrow$ and hence all formulas as well. $\Box$

We see the class $[\Box p]_{\equiv}$ behaves as $\mathbf{0}$ of the algebra $\mathbf{Grz}^{\leq n}/_{\equiv}$, whereas $[p \rightarrow p]_{\equiv}$ as $\mathbf{1}$.

Lemma 6. *Every algebra* $(\mathbf{Grz}^{\leq n}/_{\equiv}, \mathbf{1}, \rightarrow)$ *is an implication algebra including* $\mathbf{0} = [\Box p]_{\equiv}$.

Proof. The implication $\rightarrow$ is just classical one. For details see [2]. $\Box$

It is well known (see again [2]) that every implication algebra with the zero element might be extended to a Boolean algebra by introducing new operations $\vee, \sim, \wedge$ defined in the conventional way. Hence we have:

Corollary 7. *Every algebra* $(\mathbf{Grz}^{\leq n}/_{\equiv}, \mathbf{1}, \rightarrow, \vee, \wedge, \sim)$ *is a Boolean algebra.*

It is also obvious that:

Lemma 8. *Every algebra* $(\mathbf{Grz}^{\leq n}/_{\equiv}, \mathbf{1}, \rightarrow, \vee, \wedge, \sim, \Box)$ *is a modal algebra.*

3 Reduction of models

The main task of this section is to characterize the quotient algebras $\mathbf{Grz}^{\leq n}/_{\equiv}$. The crucial point in this attempt will be a theorem allowing reduction of finite reflexive and transitive tree to the linear ordered frame consisting of n points. First, we recall some definitions of the needed notions and some theorems as well (for details see [1] or [3]).
The length of formula is defined in the conventional way:

Definition 9.

$$\begin{aligned} l(p) &= 1 \\ l(\Box\phi) &= 1 + l(\phi) \\ l(\phi \rightarrow \psi) &= l(\phi) + l(\psi) + 1 \end{aligned}$$

Definition 10. *A point* x *in a frame* $\mathfrak{F}$ *is of depth* d *iff the subframe generated by* x *is of depth* d.

Definition 11. *Suppose we have two frames* $\mathfrak{F} =< W, R >$ *and* $\mathfrak{G} =< U, S >$. *A map* f *from* W *onto* U *is called a p-morphism if the following conditions hold for every* $x, y \in W$*:*

$$xRy \quad \text{implies} \quad f(x)Sf(y) \tag{4}$$

$$f(x)Sf(y) \quad \text{implies} \quad \exists_{z\in W}(xRz \wedge f(z) = f(y)) \tag{5}$$

Definition 12. *A p-morphism* f *of* $\mathfrak{F}$ *to* $\mathfrak{G}$ *is a p-morphism from a model* $\mathfrak{M} =< \mathfrak{F}, V_1 >$ *to a model* $\mathfrak{N} =< \mathfrak{G}, V_2 >$ *if for every variable* p*, for every point* $x \in \mathfrak{F}$*:*

$$(\mathfrak{M}, x) \models p \quad \text{iff} \quad (\mathfrak{N}, f(x)) \models p \tag{6}$$

We say in that case the model $\mathfrak{M}$ is reducible to the model $\mathfrak{N}$. It is well known (see [1], pp.31) that

Theorem 13. *If f is a p-morphism from a model $\mathfrak{M} =< \mathfrak{F}, V_1 >$ to a model $\mathfrak{N} =< \mathfrak{G}, V_2 >$ then for every formula φ and for every point $x \in \mathfrak{F}$:*

$$(\mathfrak{M}, x) \models \varphi \;\; \textit{iff} \;\; (\mathfrak{N}, f(x)) \models \varphi \tag{7}$$

Now, we are ready to start the reduction of trees.

Lemma 14. *Let $\mathfrak{F}_1 =< W^{\leq n} \cup \{x'\}, R >$ and $\mathfrak{F}_2 =< W^{\leq n}, R >$ be two reflexive and transitive trees with the length n, where x' is the point of depth 1 such that $(\mathfrak{M}_1, x') \models p$. Let $\mathfrak{M}_1 =< \mathfrak{F}_1, V_1 >$ and $\mathfrak{M}_2 =< \mathfrak{F}_2, V_2 >$ and the valuations V_1 and V_2 of p do not differ in $\mathfrak{F}_1$ and $\mathfrak{F}_2$ at the same points.*
For any $\alpha \in \mathcal{F}^{\{\rightarrow,\Box\}}$, for any $x_i \in W^{\leq n}$ the following equivalence holds:

$$(\mathfrak{M}_1, x_i) \models \alpha \;\; \textit{iff} \;\; (\mathfrak{M}_2, x_i) \models \alpha \tag{8}$$

Proof. We use double induction on the depth of points and length of formula. At the points $x_i \notin x' \downarrow$ (8) holds trivially. Let $(x_1, x_2, ..., x_k, x')$, $k < n$ be any chain of points from $W^{\leq n}$. For $i = 0$ and $\alpha = p$ we have $x_{k-i} = x_k$ and it is trivial that (8) holds. Suppose (8) holds at x_k for α with the length $\leq t$. We show it holds for $t + 1$. We have two cases:

1. Let $\alpha = \alpha_1 \rightarrow \alpha_2$ and $(\mathfrak{M}_1, x_k) \not\models \alpha$. Then $(\mathfrak{M}_1, x_k) \models \alpha_1$ and $(\mathfrak{M}_1, x_k) \not\models \alpha_2$. From inductive hypothesis we have $(\mathfrak{M}_2, x_k) \models \alpha_1$ and $(\mathfrak{M}_2, x_k) \not\models \alpha_2$ and hence $(\mathfrak{M}_2, x_k) \not\models \alpha$. The proof of reverse implication is analogous.

2. Let $\alpha = \Box\alpha_1$ and $(\mathfrak{M}_1, x_k) \not\models \Box\alpha_1$.

   (a) Suppose it is because $(\mathfrak{M}_1, x_k) \not\models \alpha_1$. From inductive hypothesis we have $(\mathfrak{M}_2, x_k) \not\models \alpha_1$. Then $(\mathfrak{M}_1, x_k) \not\models \Box\alpha_1$.

   (b) Suppose we have $(\mathfrak{M}_1, x_k) \models \alpha_1$ and for some $x_l \in x_k \uparrow$ holds $(\mathfrak{M}_1, x_l) \not\models \alpha_1$. But it is impossible because the only point x_l being the successor of x_k is x' at which every formula $\alpha \in \mathcal{F}^{\{\rightarrow,\Box\}}$ is true (it is the last point in the frame $\mathfrak{M}_1$).
   If $(\mathfrak{M}_1, x_k) \models \Box\alpha_1$ the case is obvious.

Suppose now (8) holds at points of depth $\leq i$ for every formula α. It should be shown that if (8) holds at point $x_{k-(i+1)}$ for α of length $\leq t$, then also for α of length $t + 1$. In that case the inductive step proceeds analogously to the one presented above. $\Box$

On the base of Lemma 14 we need only to consider as a models for implicational-necessitational reducts with one variable of Grzegorczyk's logic the finite reflexive and transitive trees with the last points on their branches falsifying p. It coincides with the condition of consistency of models which, in general involves $Grz^{\leq n} \neq \mathcal{F}^{\{\rightarrow,\Box\}}$.

Lemma 15. *Let $\mathfrak{M}_1 =< W, R, V_1 >$ and $\mathfrak{M}_2 =< U, S, V_2 >$ be two reflexive and transitive trees with the length n and with the last points falsifying p. Let $(y_1, y_2, ..., y_l)$, $l \leq n$ be any chain of points from U. Let the valuation in $\mathfrak{M}_2$ is defined as follows:*

$$(\mathfrak{M}_2, y_i) \models p \ \textit{ iff } \ (\mathfrak{M}_2, y_{i+1}) \not\models p \tag{9}$$

for any $y_i \in U$ and $1 \leq i \leq n-1$. Then the model $\mathfrak{M}_1$ is reducible to the model $\mathfrak{M}_2$.

Proof. We show that there is a p-morphism from $\mathfrak{M}_1$ onto $\mathfrak{M}_2$, which glues the neighbouring points if they all falsify or satisfy p. Let $(x_1, x_2, ..., x_k)$, $k < n$ be any chain of points from W and x_k is the point of depth 1. The p-morphism is defined as follows: $f(x_k) = y_l$ and $y_l \not\models p$. If $x_{k-1} \not\models p$ then $f(x_{k-1}) = y_l$. It proceeds as long as at some point x_{k-i} p is true . Then $f(x_{k-i}) = y_{l-1}$. This process is continued to the point x_1. The conditions (4),(5) hold obviously as well as the one (6).

Lemma 16. *Let $\mathfrak{M}_1 =< W, R, V_1 >$ be a reflexive and transitive tree with the length n, with the last points falsifying p and with the valuation defined as follows:*

$$(\mathfrak{M}_1, x_i) \models p \ \textit{ iff } \ (\mathfrak{M}_1, x_{i+1}) \not\models p \tag{10}$$

for any $x_i \in W$ and $1 \leq i \leq n-1$.
Let $\mathfrak{M}_2 =< U, S, V_2 >$ be a linear reflexive and transitive frame with the length n with the last point falsifying p and with the valuation defined:

$$(\mathfrak{M}_2, y_i) \models p \ \textit{ iff } \ (\mathfrak{M}_2, y_{i+1}) \not\models p. \tag{11}$$

for any $y_i \in U$ and $1 \leq i \leq n-1$.
Then the model $\mathfrak{M}_1$ is reducible to the $\mathfrak{M}_2$.

Proof. Since the model $\mathfrak{M}_2$ is linearly ordered it is simply a chain of n points $(y_1, y_2, ..., y_n)$ such that

$$y_{n-2t} \models p \tag{12}$$

$$y_{n-(2t+1)} \not\models p \tag{13}$$

for $t \geq 0$ and such that $1 \leq n - 2t \leq n$ and $1 \leq n - (2t + 1) \leq n$.
Let x_{k-i} be any point of depth i from W. The p-morphism will be glue the points of the same depth in $\mathfrak{M}_1$ and is defined as follows:

$$f(x_{k-i}) = y_{k-i} \tag{14}$$

for $1 \leq k \leq n$ and $i < n$. The function defined above fulfills all the needed conditions (4), (5) and (6) to be a p-morphism. □

From the above lemma we conclude that every extension of Grzegorczyk's logic $\mathbf{Grz}^{\leq n}$ in the reducted language is characterized by the single linear frame $<(x_1, x_2, ..., x_n), R>$ with the last point falsifying p and satisfying the conditions (12) and (13). That linear frame will be signed as $\mathfrak{F}_L^{\leq n}$. Every adequate modal frame is uniquely associated with some Boolean algebra. Hence, the simplicity of $\mathfrak{F}_L^{\leq n}$ makes simple the investigation of the appropriate Tarski-Lindenbaum algebra (for details see [1]). Below we present some examples of frames and appropriate quotient algebras.

Example 17. *The diagram of the algebra $Grz^{\leq 1}/_{\equiv}$ is the following:*

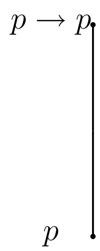

Diagram 1

Example 18. *Diagram 2 presents both the frame $\mathfrak{F}_L^{\leq 2}$ and the Tarski-Lindenbaum algebra $\mathbf{Grz}^{\leq 2}/_{\equiv}$.*

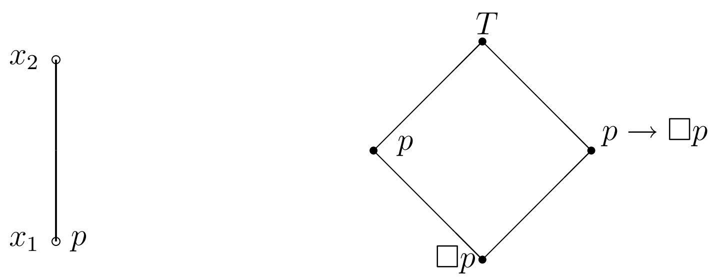

Diagram 2

Example 19. *The diagrams of $\mathfrak{F}_{Grz}^{\leq 3}$ and the Tarski-Lindenbaum algebra $\mathbf{Grz}^{\leq 3}/_{\equiv}$ are the following:*

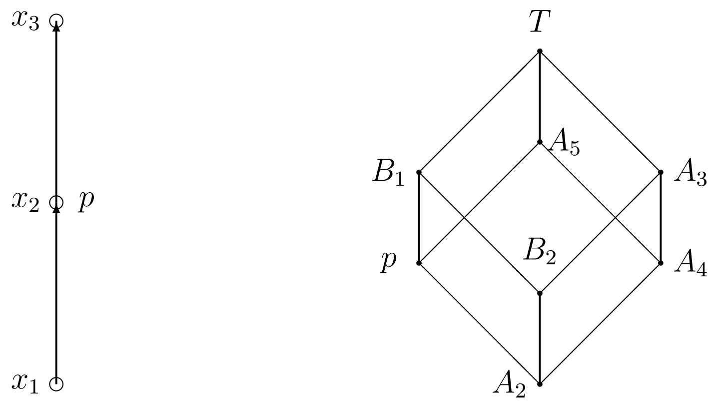

Diagram 3.

where

$$\begin{array}{rcl} A_1 & = & [p]_{\equiv} \\ A_2 & = & \Box A_1 \\ A_3 & = & A_1 \to A_2 \\ A_4 & = & \Box A_3 \\ A_5 & = & A_3 \to A_4 \\ B_1 & = & A_4 \to A_2 \\ B_2 & = & A_5 \to A_2 \end{array}$$

4 Determining the algebra $\mathbf{Grz}^{\leq n}/_{\equiv}$

The aim of this section is proving that for any $n \in \mathbb{N}$ the algebra $\mathbf{Grz}^{\leq n}/_{\equiv}$ is finite and has exactly 2^n elements. We start with analyzing the appropriate linear frame $\mathfrak{F}_L^{\leq n}$. Let us define by induction the following formulas:

Definition 20.

$$A_1 = p, \qquad A_{2n} = \Box A_{2n-1}, \qquad A_{2n+1} = A_{2n-1} \to A_{2n}, \quad \textit{for} \quad n \geq 1.$$

Lemma 21. *Let $\mathfrak{F}_L^{\leq n}$ be the linear frame for $\mathbf{Grz}^{\leq n}$. For any $k = 0, ..., n-1$:*

$$x_{n-k} \uparrow \ \models \ A_{k'} \ \textit{for any} \ k' \geq 2k+3. \tag{15}$$

Proof. By induction on k. If $k = 0$ then the point x_n is the last point in the chain $(x_1, ..., x_n)$. From Lemma 14, $x_n \not\models p$ and hence $x_n \not\models \Box p$. This gives us $x_n \models A_3$. It is easy to notice that $x_n \models A_{k'}$ for $k' \geq 3$.
Assuming (15) to hold for points of depth $\leq k$, we have $x_{n-k} \uparrow \models A_{k'}$ for $k' \geq 2k+3$ and also $x_{n-k} \uparrow \models A_{2k+3}$. We will prove $x_{n-k-1} \models A_{2k+5}$. If not, then $x_{n-k-1} \models A_{2k+3}$ and $x_{n-k-1} \not\models \Box A_{2k+3}$. Hence there is a point $x' \in x_{n-k-1} \uparrow$ such that $x' \not\models A_{2k+3}$, but it is a contradiction. From inductive hypothesis we have also $x_{n-k-1} \uparrow \models A_{k'}$ for $k' \geq 2k+5$. $\Box$

Lemma 22. *Let $\mathfrak{F}_L^{\leq n}$ be the linear frame for* $\mathbf{Grz}^{\leq n}$*. Then*

$$\begin{array}{rll} x_{n-2k} & \models & A_{4k'+3} \quad \text{and} \quad x_{n-2k} \not\models A_{4k'+1} \\ & & \text{for any } k' \geq k \text{ and } 1 \leq n-2k \leq n, \end{array} \tag{16}$$

$$\begin{array}{rll} x_{n-(2k-1)} & \models & A_{4k'+1} \quad \text{and} \quad x_{n-(2k-1)} \not\models A_{4k'+3} \\ & & \text{for any } k' \geq k \text{ and } 1 \leq n-(2k-1) \leq n, \end{array} \tag{17}$$

Proof. We use double induction with respect to the k and k'. Let $k = 0$. Then $k' = 0$ and $x_n \not\models p$ and $x_n \models A_3$. We obtained (16). If $k = 1$ then $x_{n-1} \models p$, $x_{n-1} \not\models \Box p$ and hence $x_{n-1} \not\models A_3$. We obtained (17). Assume (16) and (17) hold for some k. We show they hold for $k+1$. Assume now they hold for some $k' \geq k$ and take $k'+1$. Let us consider the formula $A_{4k'+7} = A_{4k'+5} \to \Box A_{4k'+5}$. We will prove $x_{n-(2k+2)} \not\models A_{4k'+5}$. We know that $x_{n-(2k+2)} \models A_{4k'+3}$ and $x_{n-(2k+2)} \not\models \Box A_{4k'+3}$ because $x_{n-(2k-1)} \not\models A_{4k'+3}$. So, $x_{n-(2k+2)} \models A_{4k'+7}$ and also $x_{n-(2k+2)} \not\models A_{4k'+5}$. The proof of (17) proceeds similarly. $\Box$

Corollary 23. *Let $\mathfrak{F}_L^{\leq n}$ be the linear frame for* $\mathbf{Grz}^{\leq n}$*. For any $k = 0, 1, ..., n-1$:*

$$\max\{k' : x_{n-k} \not\models A_{2k'+1}\} = k. \tag{18}$$

Corollary 24. *Let $\mathfrak{F}_L^{\leq n}$ be the linear frame for* $\mathbf{Grz}^{\leq n}$*. For any $k = 0, 1, ..., n-1$:*

$$x_{n-k} \not\models A_{2k'+5} \to A_{2k'+1} \text{ iff } k' = k. \tag{19}$$

Because considered frames are 1- generated they are also atomic (see [1], pp.270) that are frames with every point being an atom. A point x is an atom in a frame if there is a formula ϕ being true only at that point.

Theorem 25. *The following classes are atoms in every linear frame $\mathfrak{F}_L^{\leq n}$:*

$$(A_{2k+5} \to A_{2k+1}) \to A_2 \quad \text{for} \quad k = 0, 1, ..., n-1$$

Proof. In the linear frame $\mathfrak{F}_L^{\leq n}$ for any $k \leq n$ we have: $x_k \not\models A_2$. So we see that the point x_{n-k} is the only point at which the formula $(A_{2k+5} \to A_{2k+1}) \to A_2$ is true.

Corollary 26. *Every algebra* $\mathbf{Grz}^{\leq n}/_{\equiv}$ *consists of 2^n equivalence classes generated by n atoms.*

In the picture below the linear frame $\mathfrak{F}_L^{\leq n}$ with listed atoms is presented.

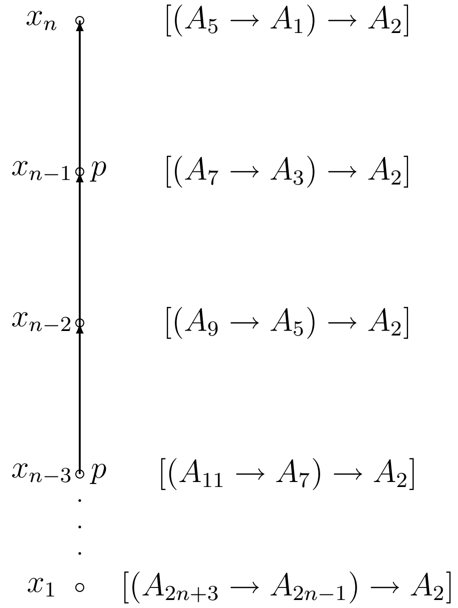

Diagram 3.

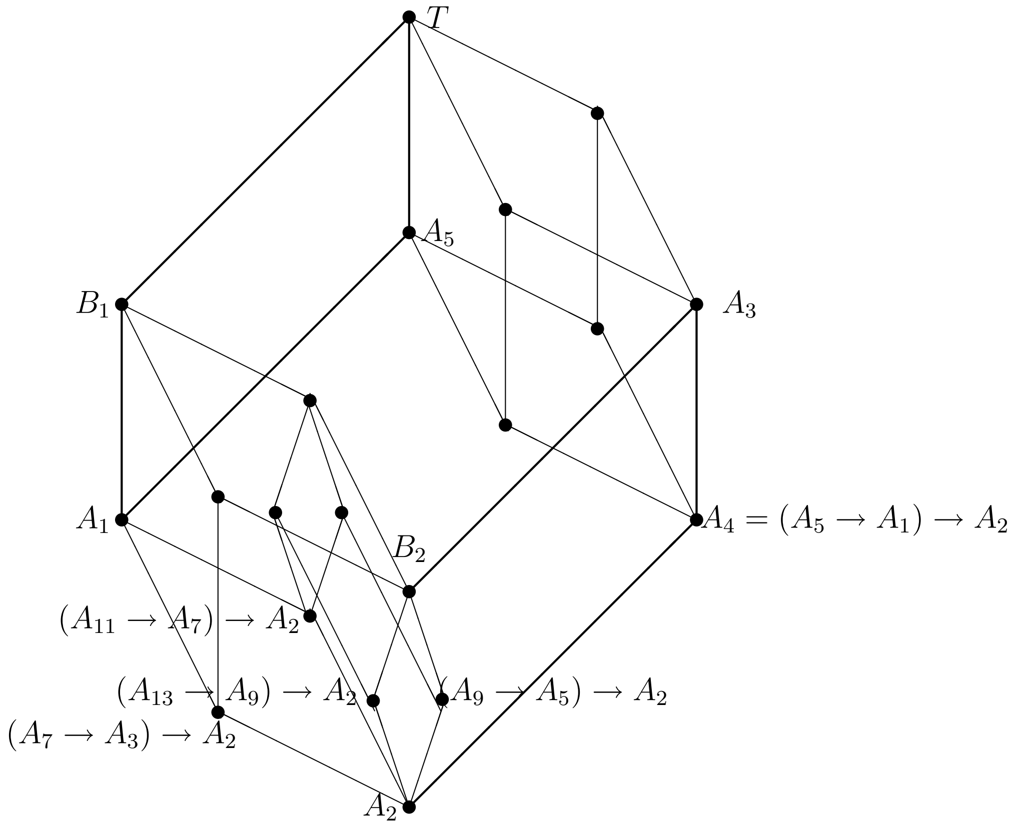

Diagram 4.

Diagram 4 presents the rule of raising of the quotient algebra $\mathbf{Grz}^{\leq n}/_{\equiv}$. More exactly - the whole algebra $\mathbf{Grz}^{\leq 4}/_{\equiv}$ is drawn with a one cube being a part of $\mathbf{Grz}^{\leq 5}/_{\equiv}$. The diagram of $\mathbf{Grz}^{\leq 5}/_{\equiv}$ consists of four analogous cubes not being marked in the picture. The classes of atoms are however listed.

References

[1] A. Chagrow and M. Zakharyaschev, **Modal Logic**, Oxford Logic Guides 35.

[2] H. Rasiowa, **An Algebraic Approach to Non-classical Logics**, PWN, Warszawa (1974).

[3] K. Segerberg, **An Essay in Classical Modal Logic**, Uppsala (1971).

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